Section 3.2, Problem 52:

If p/q is a rational zero of the polynomial

$$f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2,$$

then p must be a divisor of the constant coefficient 2, and q must be a divisor of the leading coefficient, which is also 2. I.e., $p = \pm 1, \pm 2$ and $q = \pm 1, \pm 2$, which means that the *possible* rational zeros are

$$\frac{p}{q} = \pm 1,\ \pm 2 \ \text{ and } \ \pm\frac{1}{2}.$$

Now we test the candidates:

$$f(1) = 0\ \checkmark,\ f(-1) = -2,\ f(2) = 10,\ f(-2) = 18,\ f(1/2) = \frac{7}{4} \ \text{ and } \ f(-1/2) = 0\ \checkmark.$$

This means that $x = 1$ and $x = -1/2$ are the rational zeros of $f(x)$.

Next we use this information to factor $f(x)$. We know that $(x-1)$ and $\left(x + \frac{1}{2}\right)$ are both factors of $f(x)$, so $f(x) = (x-1)\left(x+\frac{1}{2}\right)g(x)$, and since f has degree 4, it follows that g must have degree 2, i.e., $g(x) = ax^2 + bx + c$. This means that

$$\begin{aligned} 2x^4 - x^3 - 5x^2 + 2x + 2 &= (x-1)\left(x+\tfrac{1}{2}\right)(ax^2+bx+c) \\ &= \left(x^2 - \tfrac{1}{2}x - \tfrac{1}{2}\right)(ax^2+bx+c) \\ &= ax^4 + \left(b - \frac{1}{2}a\right)x^3 + \left(c - \tfrac{1}{2}a - \tfrac{1}{2}b\right)x^2 - \tfrac{1}{2}(b+c)x - \tfrac{1}{2}c. \end{aligned}$$

Comparing coefficients, of the two degree four polynomials, we have

$$a = 2,\ b - \tfrac{1}{2}a = -1,\ c - \tfrac{1}{2}(a+b) = -5,\ -\tfrac{1}{2}(b+c) = 2 \ \text{ and } \ -\tfrac{1}{2}c = 2.$$

This means that $a = 2$ and $c = -4$, from which it follows that $b = 0$, so

$$g(x) = (2x^2 - 4) = 2(x^2 - 2) = 2(x - \sqrt{2})(x + \sqrt{2})$$

and $f(x)$ factors as

$$f(x) = 2(x-1)\left(x+\tfrac{1}{2}\right)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right).$$

Section 3.4, Problem 54:

The degree of the numerator $p(x) = x^4 - 16$ is 2 more than the degree of the denominator $q(x) = x^2 - 2x$, so the rational function

$$F(x) = \frac{x^4 - 16}{x^2 - 2x}$$

does *not* have an oblique or horizontal asymptote.

Next, factoring the numerator and the denominator, we have

$$x^4 - 16 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4) \ \text{ and } \ x^2 - 2x = x(x-2)$$

so the reduced form of $F(x)$ is

$$F_r(x) = \frac{\cancel{(x-2)}(x+2)(x^2+4)}{x\cancel{(x-2)}} = \frac{(x+2)(x^2+4)}{x},$$

which means that the graph $y = F(x)$ has one vertical asymptote, $x = 0$ (and a *hole* at $(2, 16)$).

Conclusion: *No horizontal or oblique asymptotes and one vertical asymptote, $x = 0$.*

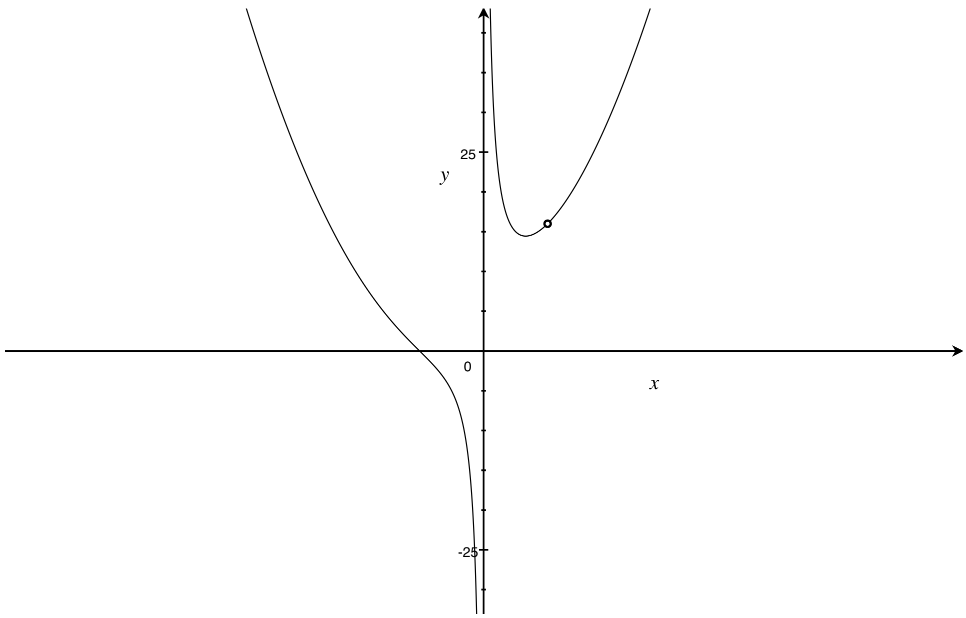

Figure 1: The graph of $F(x) = \frac{x^4-16}{x^2-2x}$

Section 3.5, Problem 8: Analyzing the graph of $R(x) = \dfrac{x}{(x-1)(x+2)}$.

Step 1. $R(x)$ is already factored above. The domain of $R(x)$ is $\{x | x \neq 1, -2\}$.

Step 2. $R(x)$ is already in lowest terms (i.e., the numerator and denominator don't have any common factors).

Step 3. $R(x)$ has one zero, at $x = 0$, i.e., $R(0) = 0$. This is both the (only) x-intercept and the y-intercept. Since x appears to an odd power in, the graph of $R(x)$ *crosses* the x-axis at this point.

Step 4. $R(x)$ has the lines $x = 1$ and $x = -2$ as vertical asymptotes.

(i) To the left of $x = -2$, $x < 0$, $x - 1 < 0$ and $x + 2 < 0$, so $R(x) < 0$ to the left of $x = -2$, and therefore $R(x) \to -\infty$ on the left side of the asymptote $x = -2$.

(ii) To the (immediate) right of $x = -2$, $x < 0$, $x - 1 < 0$ and $x + 2 > 0$, so $R(x) > 0$ to the right of $x = -2$, and therefore $R(x) \to +\infty$ on the right side of the asymptote $x = -2$.

(iii) To the (immediate) left of $x = 1$, $x > 0$, $x - 1 < 0$ and $x + 2 > 0$, so $R(x) < 0$ to the left of $x = 1$, and therefore $R(x) \to -\infty$ on the left side of the asymptote $x = 1$.

(iv) To the (immediate) right of $x = 1$, $x > 0$, $x - 1 > 0$ and $x + 2 > 0$, so $R(x) > 0$ to the right of $x = 1$, and therefore $R(x) \to +\infty$ on the right side of the asymptote $x = 1$.

Step 5. The degree of the numerator is less than the degree of the denominator, so $y = 0$ (the x-axis) is a horizontal asymptote to the graph of $R(x)$. As we already saw in Step 3., the graph intersects the x-axis at the origin and nowhere else.

Step 6. The zeros of the numerator and denominator are $x = -2$, $x = 0$ and $x = 1$, so the intervals we have to consider are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$ and $(1, \infty)$.

- $R(-3) = -\dfrac{3}{4} < 0$, so $R(x) < 0$ in $(-\infty, -2)$.
- $R(-1) = \dfrac{1}{2} > 0$, so $R(x) > 0$ in $(-2, 0)$.
- $R(1/2) = -\frac{2}{5} < 0$, so $R(x) < 0$ in $(0, 1)$.
- $R(2) = \frac{1}{2} > 0$, so $R(x) > 0$ in $(1, \infty)$.

Step 7. Comment: The horizontal asymptote is not dashed because it is also an axis.

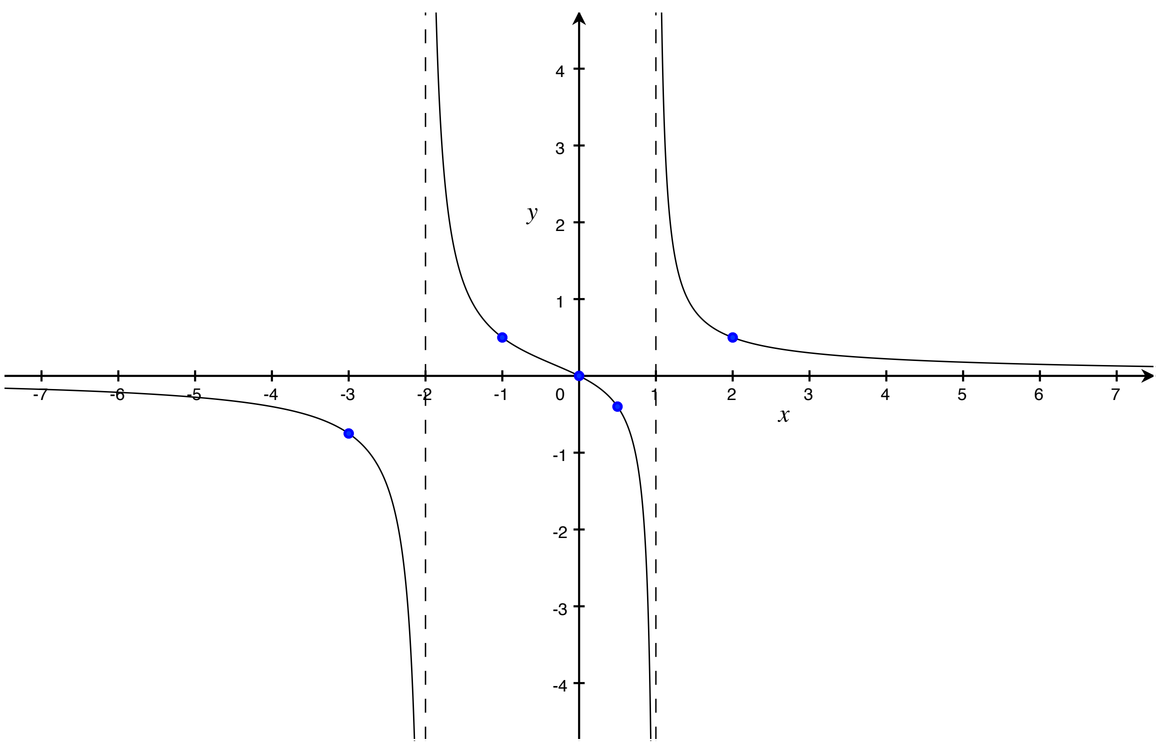

Figure 2: The graph of $R(x) = \frac{x}{(x-1)(x+2)}$