**Analyzing (another) rational function:** $f(x) = \dfrac{x^3 - 3x + 2}{2x^3 - 4x^2 - 2x + 4}$

(1a) Factoring denominator:

$$\begin{aligned} 2x^3 - 4x^2 - 2x + 4 &= 2(x^3 - 2x^2 - x + 2) \\ &= 2(x^2(x-2) - (x-2)) \\ &= 2(x^2 - 1)(x-2) = 2(x-1)(x+1)(x-2) \end{aligned}$$

(1b) Factoring numerator: *not so easy.* On the other hand, by inspection $x = 1$ is a zero of $p(x) = x^3 - 3x + 2$, which means that $(x - 1)$ is a factor of $p(x)$, so we can write

$$x^3 - 3x + 2 = (x-1)(x^2 + bx + c) = x^3 + (b-1)x^2 + (c-b)x - c$$

$$\implies c = -2 \text{ and } b = 1$$

I.e.,

$$x^3 - 3x + 2 = (x-1)(x^2 + x - 2) = (x-1)(x-1)(x+2).$$

**Factored form of $f(x)$:**

$$f(x) = \frac{x^3 - 3x + 2}{2x^3 - 4x^2 - 2x + 4} = \frac{(x-1)(x-1)(x+2)}{2(x-1)(x+1)(x-2)}.$$

(2) Domain of $f(x)$: $\{x | x \neq 1, -1, -2\}$.

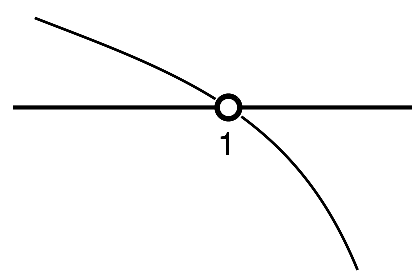
**Observation:** The factor $(x-1)$ appears both in the numerator and the denominator...

(3) Reduced form of $f(x)$:

$$f(x) = \frac{\cancel{(x-1)}(x-1)(x+2)}{2\cancel{(x-1)}(x+1)(x-2)} = \frac{(x-1)(x+2)}{2(x+1)(x-2)} = f_r(x)$$

$$\Rightarrow f(x) = f_r(x) \text{ at every point } x \neq 1. \Leftarrow$$

The graph of $f(x)$ has a '*hole*' at $x = 1$:



**Zeros.** The zeros of $f_r(x)$ are $x = 1$ and $x = -2$, but $f(x)$ is not defined at $x = 1$, so technically, $f(x)$ has only one zero, at $x = -2$. (We do consider $x = 1$ as a point where $f(x)$ may change sign, however).

**Vertical asymptotes.** These *may* occur at points where the function is not defined. There are three points to consider.

(*) At $x = -1$, the numerator of $f(x)$ is $4 \neq 0$ while the denominator is 0, so the line $x = -1$ is a vertical asymptote.

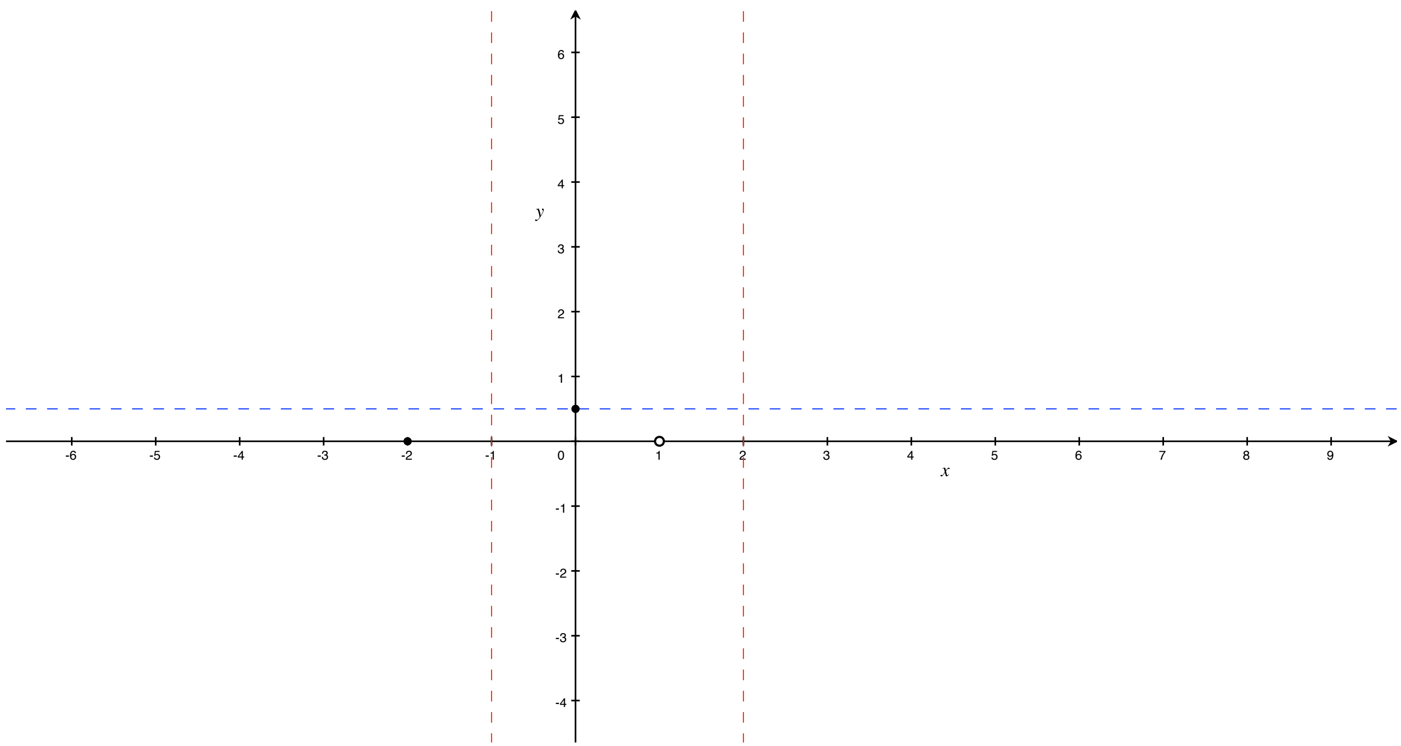
(*) At $x = 1$, the reduced from $f_r(x)$ is defined at $x = 1$ ($f_r(1) = 0$), so there is no vertical asymptote there — we already know that there is a hole in the graph at this point.

(*) At $x = 2$, the numerator of $f(x)$ is $2 \neq 0$ and the denominator is 0, so the line $x = 2$ is a vertical asymptote.

**Horizontal asymtote.** The numerator and denominator of $f_r$ have the same degree, 2, so there is a horizontal asymptote. If $|x|$ is large, then

$$f_r(x) = \frac{x^2 + x - 2}{2x^2 - 2x - 4} \approx \frac{x^2}{2x^2} = \frac{1}{2},$$

so the horizontal asymptote is the line $y = \frac{1}{2}$.




Asymptotes, zeros, $y$-intercept $(= 1/2)$ and the hole at $x = 1$.

**Signs.**

The sign of $f(x)$ can only change at: $-2$ (a zero); $-1$ (vertical asymptote); 1 (hole in the graph); 2 (vertical asympote).

$\Rightarrow$ Sample $f_r(x)$ in the intervals...

(*) $(-\infty, -2)$: $f_r(-3) = 4/20 > 0$

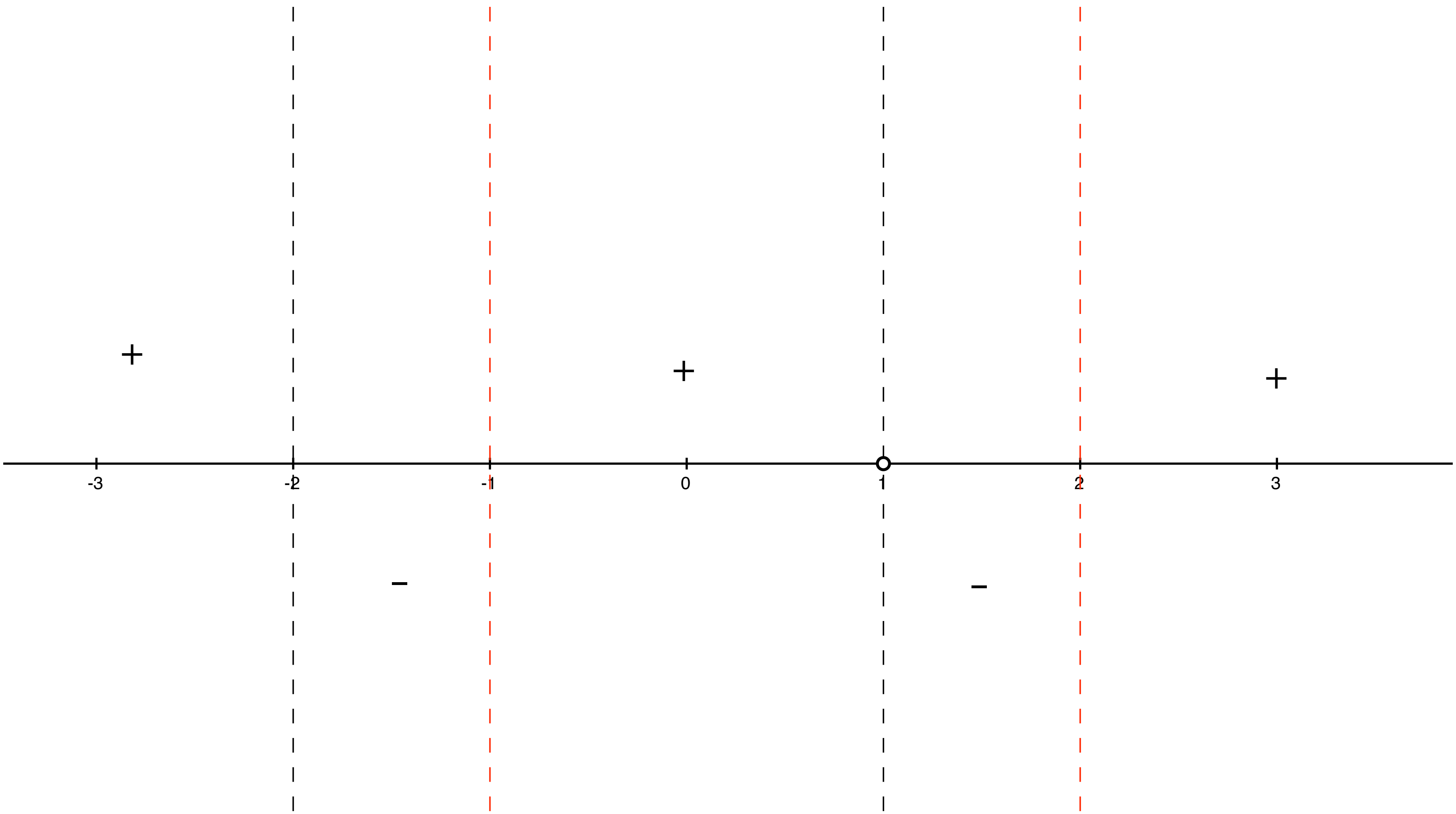
(*) $(-2, -1)$: $f_r(-1.5) = (-1.75)/2.5 < 0$

(*) $(-1, 1)$: $f(0) = 1/2 > 0$

(*) $(1, 2)$: $f_r(1.5) = 1.75/(-2.5) < 0$

(*) $(2, \infty)$: $f_r(100) \approx 1/2 > 0$.

# Signs in each interval



Putting it all together....