Rational functions

Definition: A rational function is the quotient of two polynomials:

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0} = \frac{p(x)}{q(x)}.$$

Examples:

$$f(x) = \frac{3x+2}{4x^3+5x^2-1}, \quad g(x) = \frac{x^2+x-2}{5x+3} \text{ and } h(x) = \frac{2x^2+1}{3x^2+4x+1}.$$

Important characteristics:

(a) Domain of definition.

(a) $\Rightarrow$ behavior near points where the function is undefined.

(b) Zeros.

(a) and (b) $\Rightarrow$ intervals where the function is positive and negative.

(c) End behavior (as $x \to \pm\infty$).

Domain of defintion.

(*) Polynomials are defined on the entire real line.

(*) Quotients are defined wherever the denominator is ***not zero***.

Conclusion: The rational function $y = \frac{p(x)}{q(x)}$ is defined at all points where $q(x) \neq 0$.

Example. Find the domain of

$$h(x) = \frac{2x^2 + 1}{3x^2 + 4x + 1}.$$

(*) Zeros of $3x^2 + 4x + 1 = (3x + 1)(x + 1)$: $x_1 = -1$ and $x_2 = -1/3$.

$$\Rightarrow \text{Domain}(h(x)) = \{x | x \neq -1 \text{ and } x \neq -1/3\}.$$

Behavior near points where the rational function is undefined.

(*) If $q(a) = 0$, so $f(x) = p(x)/q(x)$ is undefined at $x = a$, then $f(x)$ will <u>*usually*</u> tend to $\pm\infty$ as x gets closer to a.

Example. The function $h(x) = \dfrac{2x^2+1}{3x^2+4x+1}$ is not defined at $x = -1$.

(*) The numerator, $2x^2 + 1 > 0$ for all x.

(*) The denominator, $3x^2 + 4x + 1 > 0$ if $x < -1$ or $x > -1/3$

$\Rightarrow h(x) > 0$ if $x < -1$ or $x > -1/3$

(*) $3x^2 + 4x + 1 < 0$ if $-1 < x < -1/3$.

$\Rightarrow h(x) < 0$ if $-1 < x < -1/3$.

Conclusions:

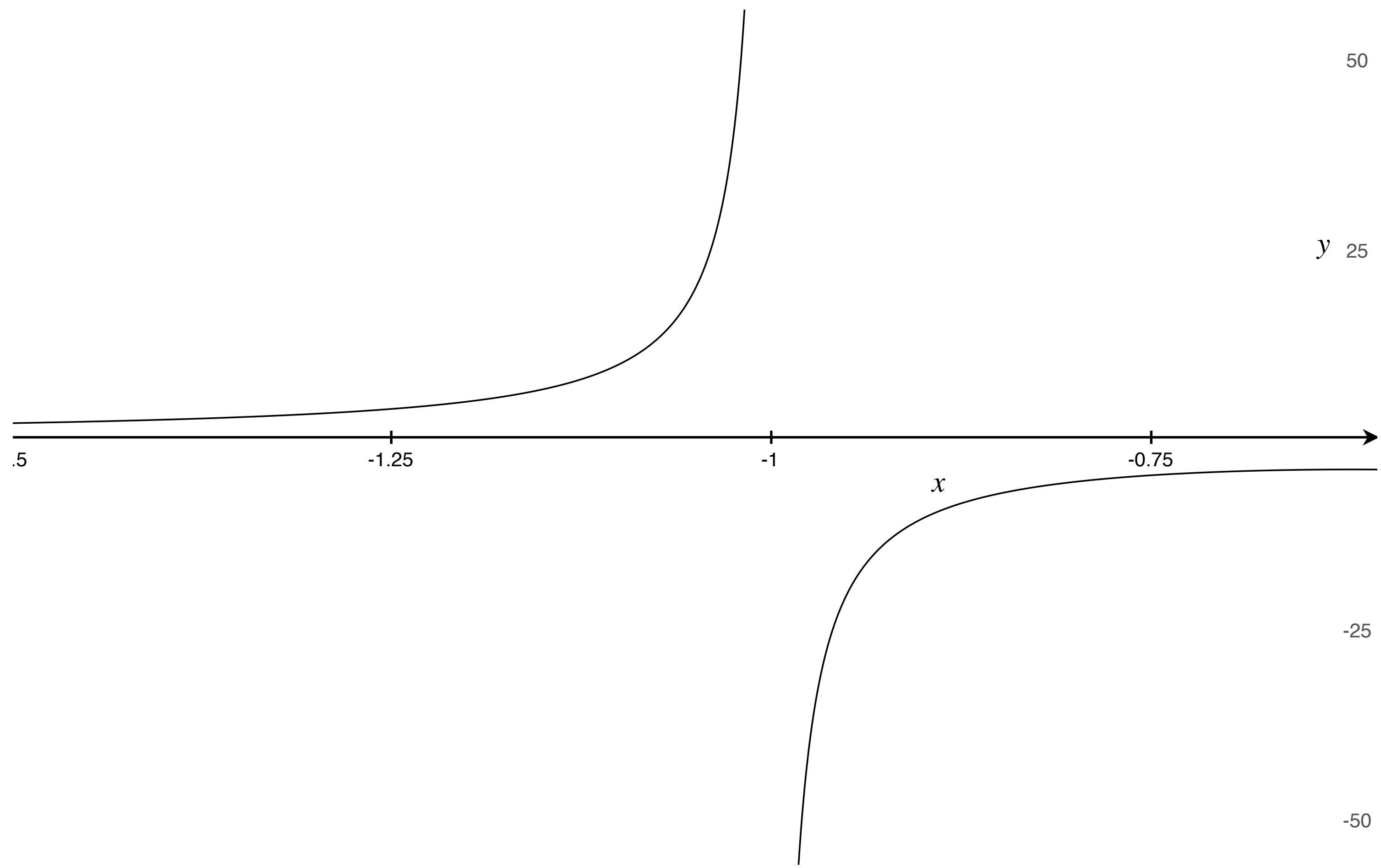
(*) If $x \approx -1$ and $x < -1$, then $h(x) > 0$ and $|h(x)|$ is very large

$$\Rightarrow h(x) \to +\infty \text{ to the left of } -1.$$

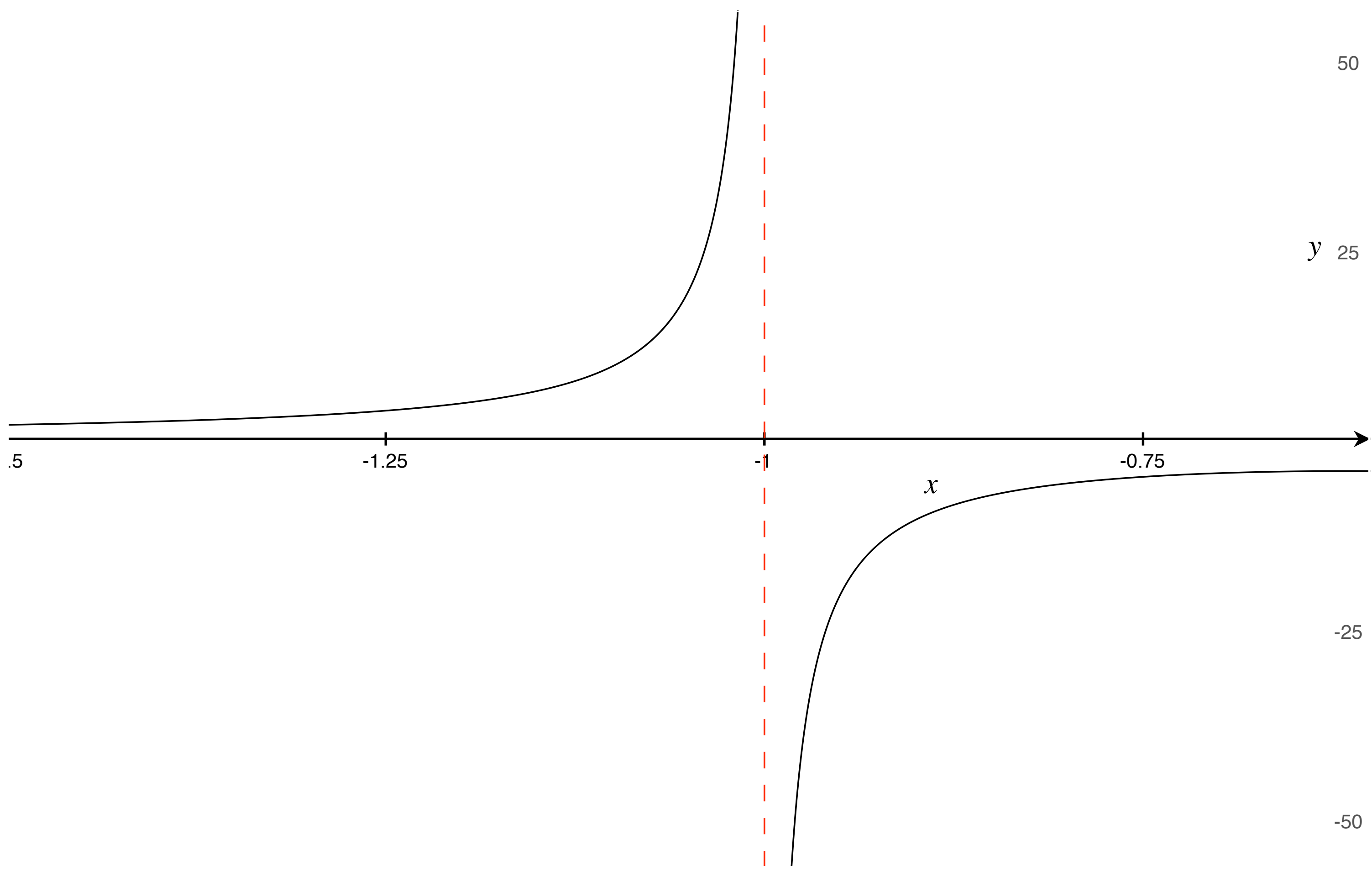
(*) If $x \approx -1$ and $x > -1$, then $h(x) < 0$ and $|h(x)|$ is very large

$$\Rightarrow h(x) \to -\infty \text{ to the right of } -1.$$

In the vicinity of $x = -1$, the graph of $y = h(x)$ looks like this:



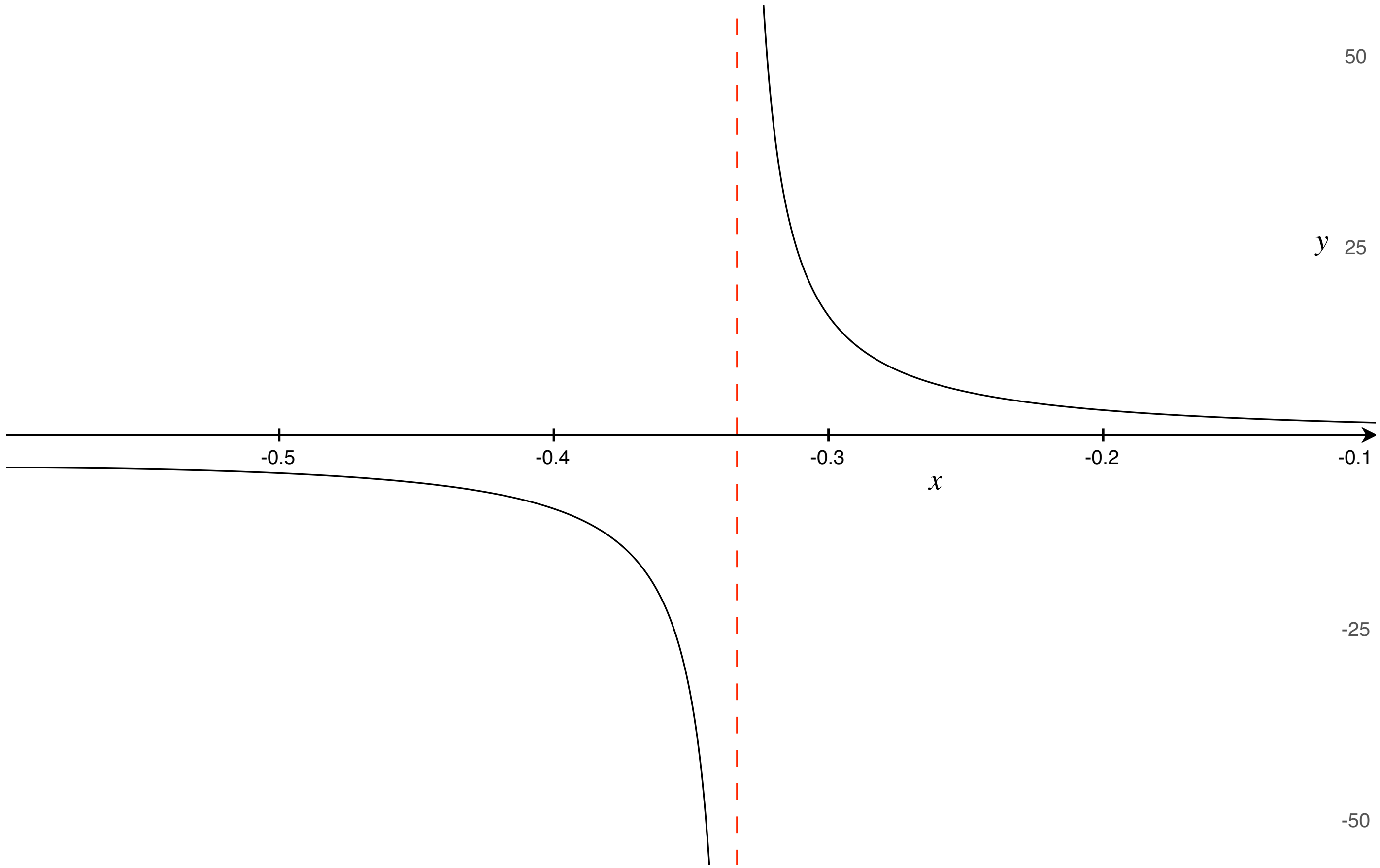
In the vicinity of $x = -1$, the graph of $y = h(x)$ looks like this:



The line $x = -1$ is called a ***vertical asymptote*** to the graph $y = h(x)$.

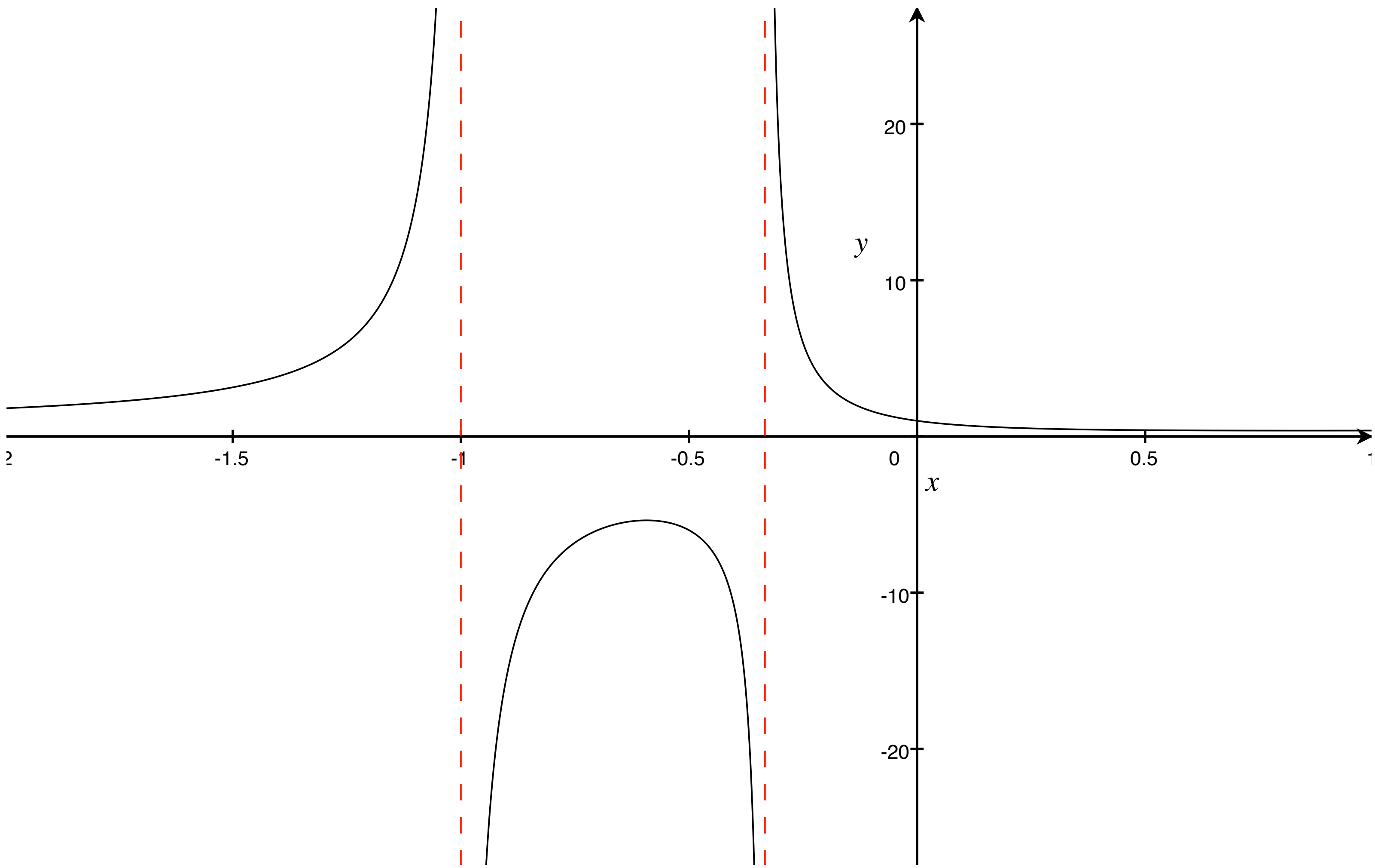
(*) The graph of a rational function cannot *cross* a vertical asymptote.

A similar analysis shows that $h(x) \to +\infty$ to the right of $-1/3$ and $h(x) \to -\infty$ to the left of $-1/3$. In the vicinity of $x = -1/3$, the graph of the function $y = h(x)$ looks like this:



The line $x = -1/3$ is another vertical asymptote to the graph $y = h(x)$.

Zooming out a little, we see that in an interval that includes both -1 and $-1/3$, the graph of $y = h(x)$ looks something like this:



Zeros of rational functions.

Fact: A simple quotient $\frac{A}{B} = 0$ if and only if $A = 0$ (and $B \neq 0$).

Conclusion: The zeros of the function $f(x) = \frac{p(x)}{q(x)}$ are the same as the zeros of the polynomial $p(x)$.

Example. The polynomial $p(x) = x^2 + 1$ has no (real) zeros, so $h(x) = \frac{x^2 + 1}{3x^2 + 4x + 1}$ has no zeros.

Example. The polynomial $p(x) = x^2 + x - 2 = (x + 2)(x - 1)$ has zeros at $x = 1$ and $x = -2$, so the rational function $g(x) = \frac{x^2 + x - 2}{5x + 3}$ has zeros at $x = 1$ and $x = -2$.

(*) $y = g(x)$ is also *not* defined at $x = -0.6$ (where $5x + 3 = 0$), and has a vertical asymptote there.

Fact: A rational function $f(x)$ can only ***change sign***, $(+) \to (-)$ or $(-) \to (+)$, (i) at a zero or (ii) at a point where it is undefined.

$\Rightarrow$ To determine where the rational function $f(x)$ is positive and where it is negative...

(i) List the zeros of $f(x)$ and points where $f(x)$ is undefined, in ascending order,

$$a_1 < a_2 < \cdots < a_k.$$

(ii) Sample the function in each of the intervals

$$(-\infty, a_1),\ (a_1, a_2), \ldots, (a_{k-1}, a_k),\ (a_k, \infty).$$

The sign of $f(x)$ at the sampled point in each interval is the sign of $f(x)$ throughout that interval.

Example. The zeros of the function $g(x) = \dfrac{x^2 + x - 2}{5x + 3}$ are -2 and 1, and the function is undefined at -0.6, so we test $g(x)$ in the intervals

$$(-\infty, -2),\ (-2, -0.6),\ (-0.6, 1)\ \text{ and }\ (1, \infty)$$

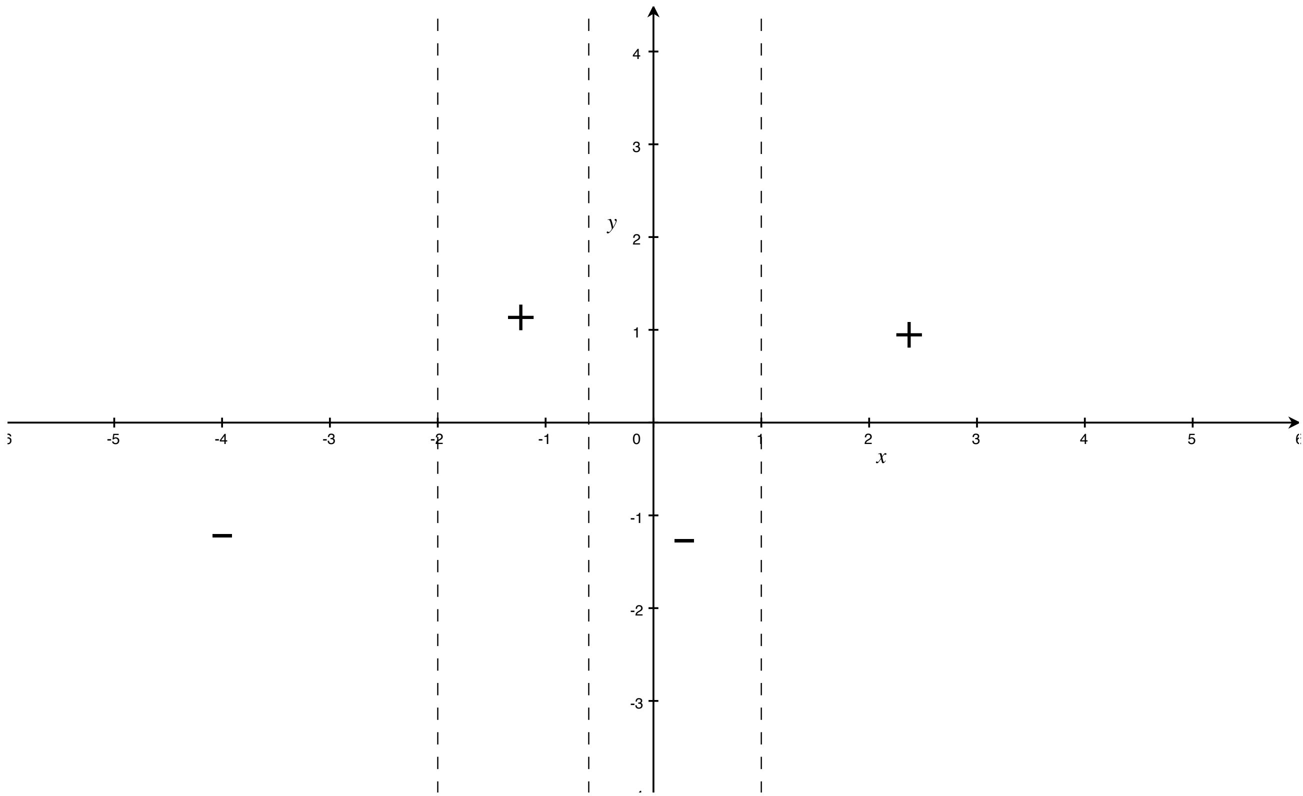
(*) $g(-3) = 4/(-48) < 0$, so $g(x) < 0$ in $(-\infty, -2)$;

(*) $g(-1) = (-2)/(-8) > 0$, so $g(x) > 0$ in $(-2, -0.6)$;

(*) $g(0) = -2/3 < 0$, so $g(x) < 0$ in $(-0.6, 1)$;

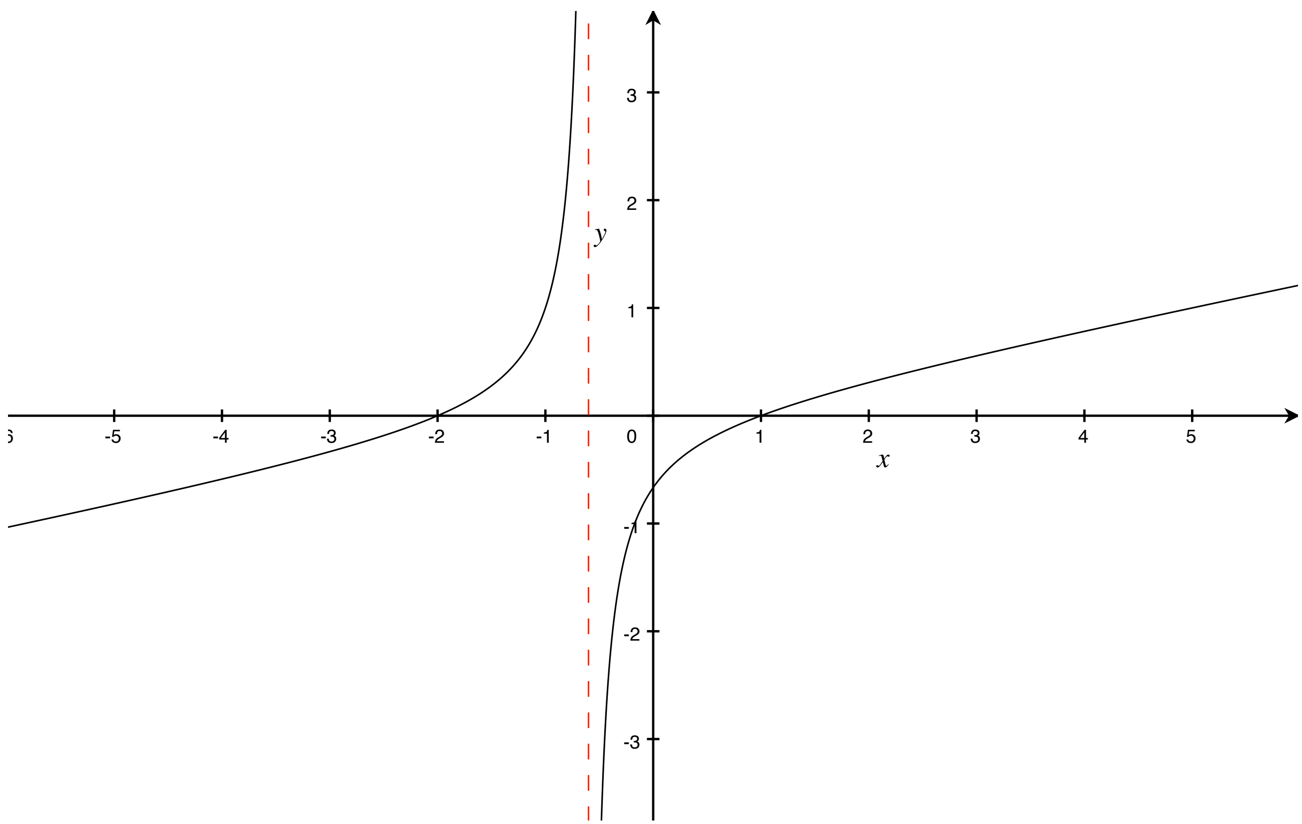
(*) $g(2) = 4/13 > 0$, so $g(x) > 0$ in $(1, \infty)$

The sign information for $g(x)$ is displayed on the next page.



y
x
+
+
−
−

Remembering that $g(x)$ has a vertical asymptote at $x = -0.6$, using the zeros, the signs above and the y-intercept $= -2/3$ for good measure, we can sketch the 'middle' of the graph $y = \dfrac{x^2 + x - 2}{5x + 3}$:



For this function, the only thing left to analyze is the behavior as $x \to \pm\infty$, the 'end-behavior of $g(x)$.

End behavior.

The 'end behavior' of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ (what the function does as $x \to \infty$ or $x \to -\infty$) depends on the degrees of $p(x)$ and $q(x)$.

Key observations:

(a) If $k > 0$, then $x^k \to \pm\infty$ as $x \to \pm\infty$. The signs $(+/-)$ depend on the parity of k (odd/even) and whether $x \to \infty$ or $x \to -\infty$.

(b) $x^0 = 1$, so $a \cdot x^0 = a$.

(c) If $k > 0$, then $x^{-k} = \dfrac{1}{x^k} \to 0$ as $x \to \pm\infty$. The signs don't matter here.

And... drum roll...

(d) $f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}$ behaves like $(a_m/b_n)x^{m-n}$ when $|x|$ is large.

Conclusions: If $f(x) = \dfrac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0}$, then

(1) If $m < n$, then $f(x) \to 0$ as $x \to \infty$ or $x \to -\infty$.

(*) In this case the line $y = 0$ (the x-axis) is a **horizontal** asymptote to the graph $y = f(x)$.

(2) If $m = n$, then $f(x) \to \dfrac{a_n}{b_n}$ as $x \to \infty$ or $x \to -\infty$.

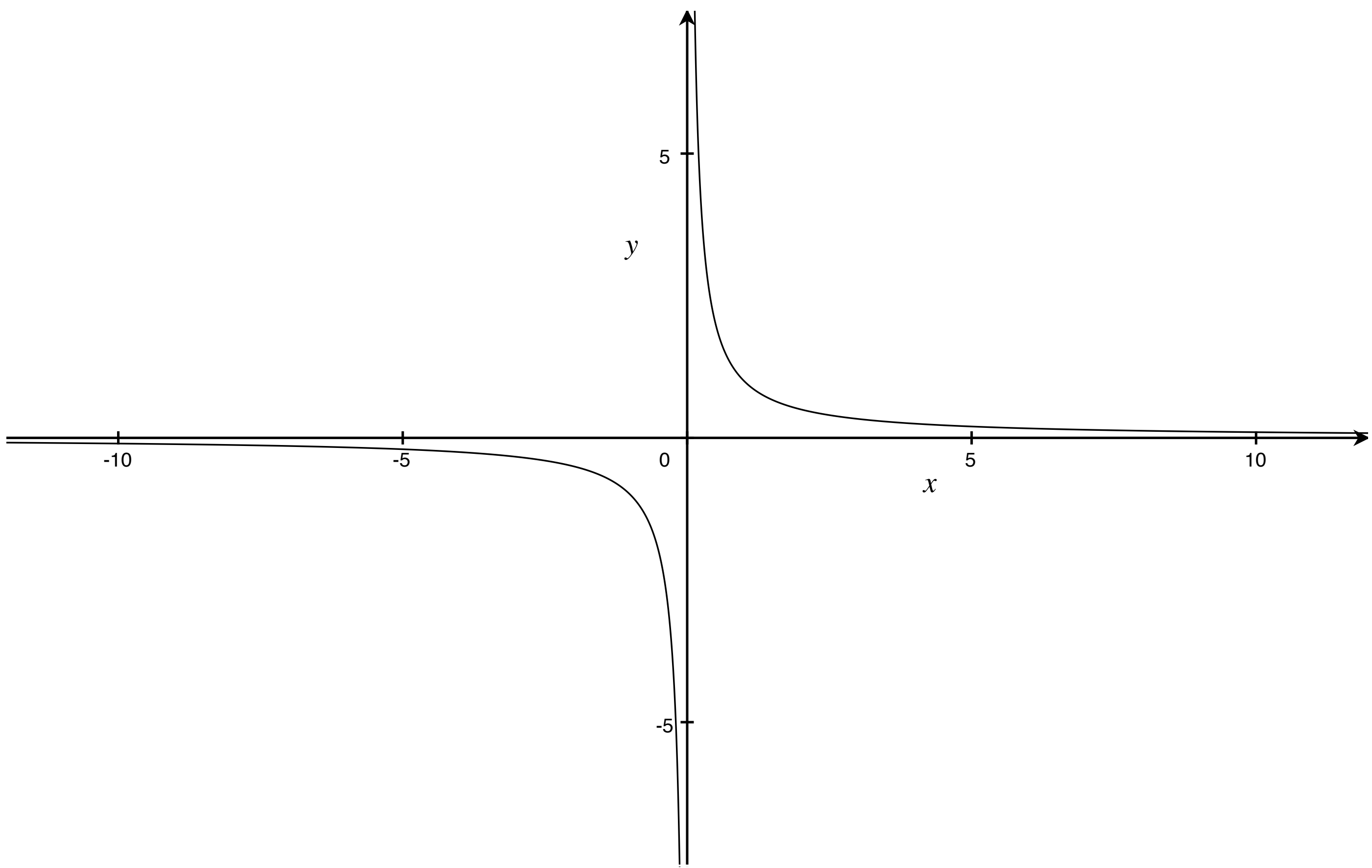
(*) In this case the line $y = a_n/b_n$ is a **horizontal** asymptote to the graph $y = f(x)$.

(3) If $m > n$, then $f(x) \to \pm\infty$ as $x \to \pm\infty$. The signs depend on the parity of $m - n$ and on a_m/b_n.

(3L) ***Special case:*** If $m - n = 1$, then $f(x)$ goes to $\pm\infty$ like a linear function.

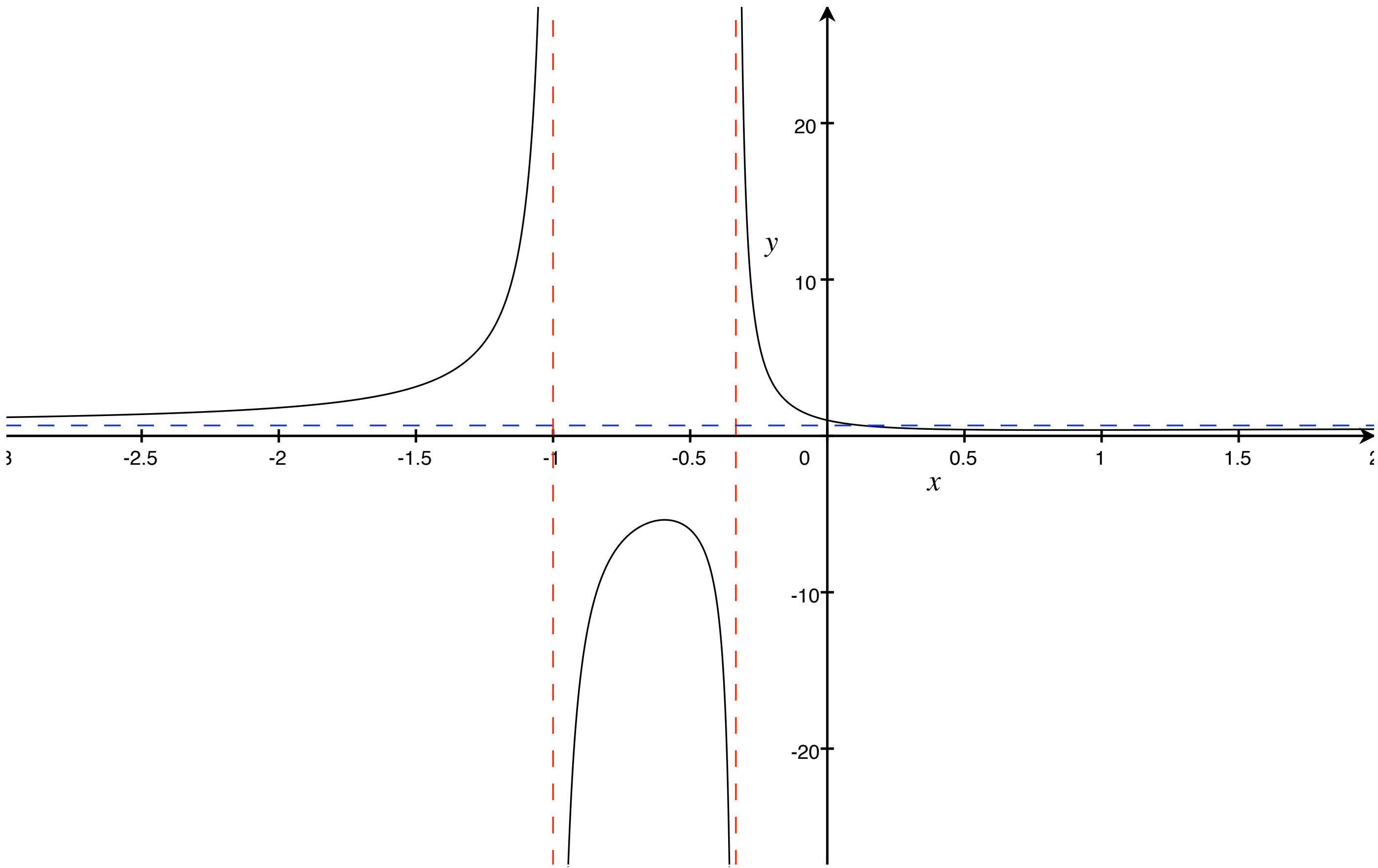
Example.

(*) The function $f(x) = \dfrac{1}{x}$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$ (the axes). Its graph:



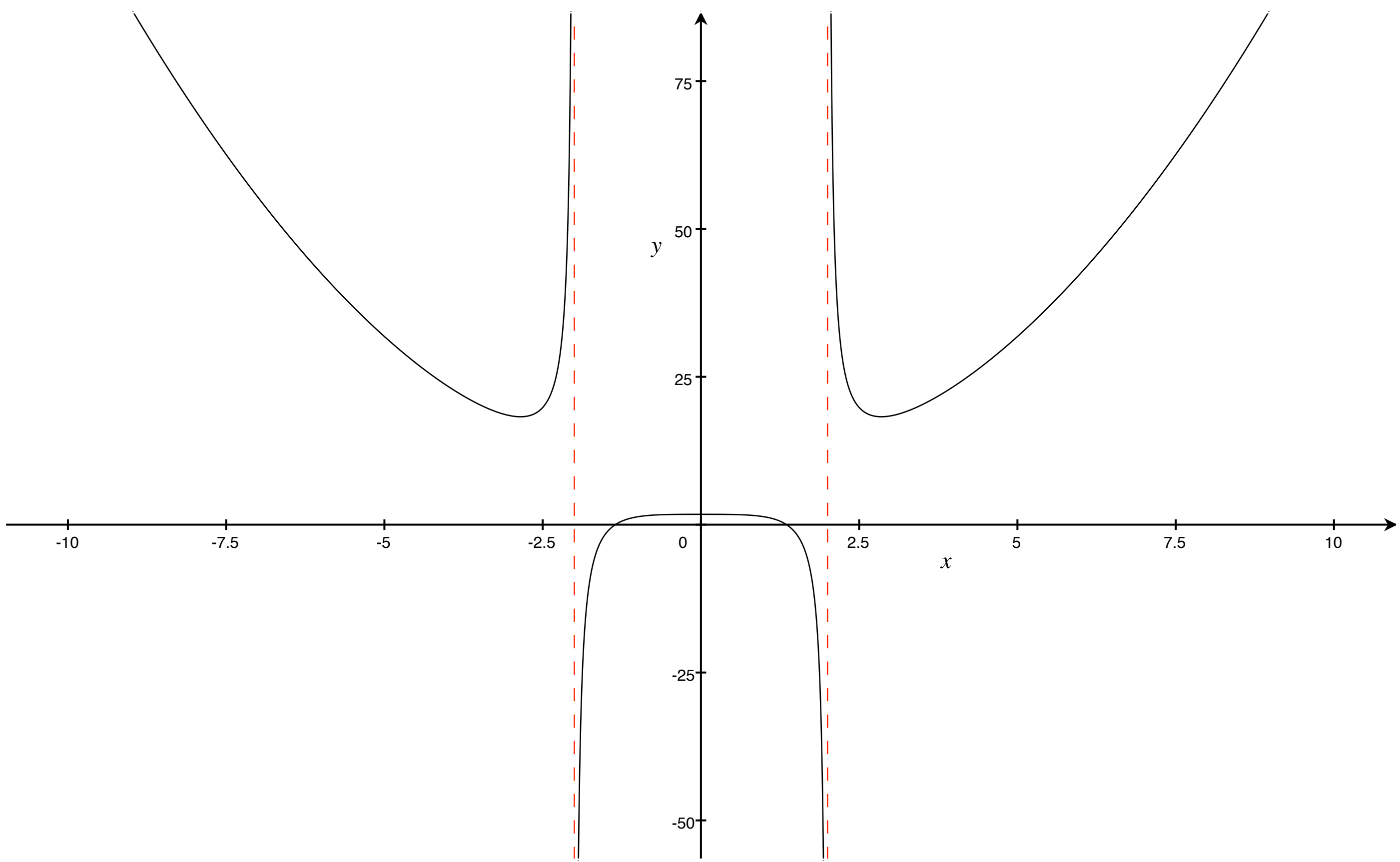
Example.

(*) The function $h(x) = \dfrac{2x^2+1}{3x^2+4x+1}$ has vertical asymptotes at $x = -1$ and $x = -1/3$ (red lines) and a horizontal asymptote at $y = 2/3$ (blue line). Its graph:



Example.

(*) The function $k(x) = \dfrac{x^4 + 2x^2 - 7}{x^2 - 4}$ has vertical asymptotes at $x = \pm 2$ and (red lines) and grows like $x^4/x^2 = x^2$ as $x \to \pm\infty$. It's graph:



Oblique asymptotes.

If $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots b_0}$, so $\deg(p) - \deg(q) = 1$, then dividing $p(x)$ by $q(x)$ with remainder (using long division), we can find...

(i) a linear function $mx + d$ (the *quotient*) and

(ii) a polynomial $c_{n-2}x^{n-2} + \cdots + c_0$ (the *remainder*)

such that

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots b_0} = (mx + d) + \frac{c_{n-2}x^{n-2} + \cdots + c_0}{b_{n-1}x^{n-1} + \cdots b_0}.$$

(*) If $|x|$ is very large, then $\dfrac{c_{n-2}x^{n-2} + \cdots + c_0}{b_{n-1}x^{n-1} + \cdots b_0} \approx 0$ (why?).

(*) So, if $|x|$ is very large, then

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots b_0} \approx mx + d.$$

The line $y = mx + d$ is called an ***oblique*** asymptote to the graph of $y = f(x)$.

Example. The numerator of $g(x) = \frac{x^2+x-2}{5x+3}$ has degree 2 and the denominator has degree 1, so $y = g(x)$ has an oblique asymptote. To find the equation of the asymptote, we divide $x^2 + x - 2$ by $5x + 3$:

$$\begin{array}{r|l} & \tfrac{1}{5}x + \tfrac{2}{25} \\ \hline 5x+3 & x^2 + x - 2 \\ & -x^2 - \tfrac{3}{5}x \\ \hline & \tfrac{2}{5}x - 2 \\ & -\tfrac{2}{5}x - \tfrac{6}{25} \\ \hline & -\tfrac{56}{25} \end{array}$$

The quotient is $\dfrac{1}{5}x + \dfrac{2}{25}$, so the oblique asymptote to $y = g(x)$ is the line

$$y = \frac{1}{5}x + \frac{2}{25}$$

The graph of $y = \dfrac{x^2 + x - 2}{5x + 3}$ with vertical asymptote $x = -0.6$ (red) and oblique asymptote $y = \frac{1}{5}x + \frac{1}{25}$ (blue):