(1) (6 pts) Find the polynomial of *least degree* whose graph is a match for the graph in Figure 1 below. Explain briefly how you found it.

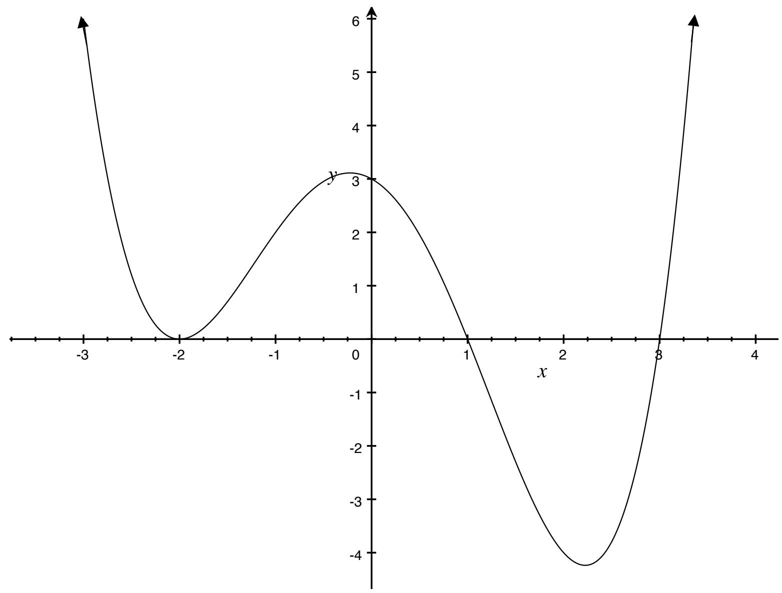


Figure 1: Graph for Problem 1

Solution:

(i) *From the x-intercepts of the graph, we know that any polynomial with this graph will have the form*

$$f(x) = g(x)(x+2)^k(x-1)^l(x-3)^m,$$

because the zeros of $f(x)$ are -2, 1 and 3. The factor $g(x)$ is a polynomial with ***no*** *real zeros.*

(ii) *The power k must be* ***even*** *and the powers l and m must be* ***odd***, *because the graph touches (but does not cross) the x-axis at $(-2, 0)$ and the graph crosses the x-axis at $(1, 0)$ and $(3, 0)$.*

(iii) *Since we want the degree of $f(x)$ to be as small as possible, we choose $k = 2$ and $l = m = 1$. We also choose $g(x)$ to be a* constant, *i.e., $g(x) = a$.*

From (i), (ii) and (iii), we conclude that $f(x) = a(x+2)^2(x-1)(x-3)$.

(iv) *Finally, we use the y-intercept, $(0, 3)$ to determine the constant a:*

$$f(0) = 3 \implies a(0-2)^2(0-1)(0-3) = 3 \implies 12a = 3 \implies a = \frac{3}{12} = \frac{1}{4},$$

so the polynomial that we seek is

$$f(x) = \frac{1}{4}(x+2)^2(x-1)(x-3) \qquad \left(= \frac{1}{4}x^4 - \frac{9}{4}x^2 - x + 3\right).$$

(2) Newton's law of cooling states that the temperature of a body $u(t)$ at time t immersed in a medium (air, water, etc.) of constant ambient temperature T, can be modeled by the function

$$u(t) = T + (u_0 - T)e^{kt},$$

where u_0 is the initial temperature of the body (at the time of immersion) and $k < 0$ is a constant.

A pizza is baked at 450° Fahrenheit. When it is done, at which point its temperature is 450°, the pizza is removed from the oven and left to cool in a room with constant ambient temperature $T = 70°$ Fahrenheit.

(a) (3 pts) After 5 minutes, the temperature of the pizza is 200°. Find the constant k for this pizza.

Solution: *By Newton's law,*

$$u(5) = T + (u_0 - T)e^{5k} \implies 200 = 70 + (450 - 70)e^{5k} \implies 130 = 380e^{5k}.$$

Dividing both sides of the equation on the right and taking natural log of both sides, allow us to solve for k:

$$\frac{130}{380} = e^{5k} \implies \ln(13/38) = 5k \implies k = \frac{1}{5}\ln(13/38) \quad (\approx -0.2145).$$

(b) (3 pts) How many minutes after it was removed from the oven will the temperature of the pizza be 90°?

Solution: *If t_1 is the time at which the pizza reaches 90°, then once again by Newton's law,*

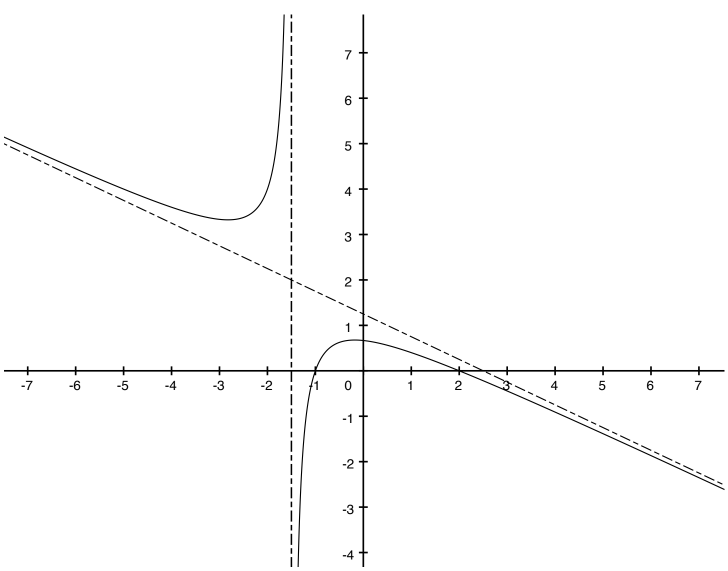
$$90 = 70 + (450 - 70)e^{kt_1} \implies 20 = 380e^{kt_1} \implies \frac{20}{380} = e^{kt_1} \implies kt_1 = \ln(1/19),$$

so

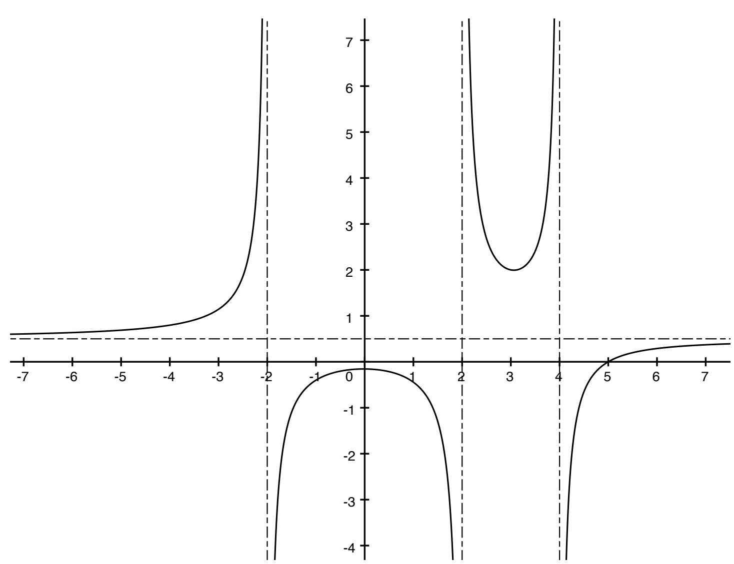
$$t_1 = \frac{\ln(1/19)}{\frac{1}{5}\ln(13/38)} \approx 13.725 \text{ minutes.}$$

(3) (6 pts) Match the following functions to one of the graphs that follow (you will have three graphs left over). *Explain **briefly** how you matched each function to its graph.*

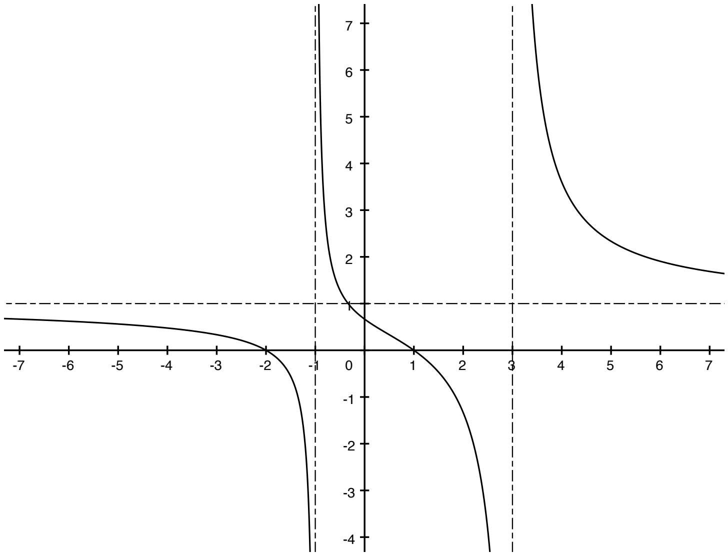
(a) $R(x) = \dfrac{x-2}{x^2-3x-4}$. (b) $H(x) = \dfrac{x^2-x-2}{2x-5}$ (c) $T(x) = \dfrac{x^2+x-2}{x^2-2x-3}$



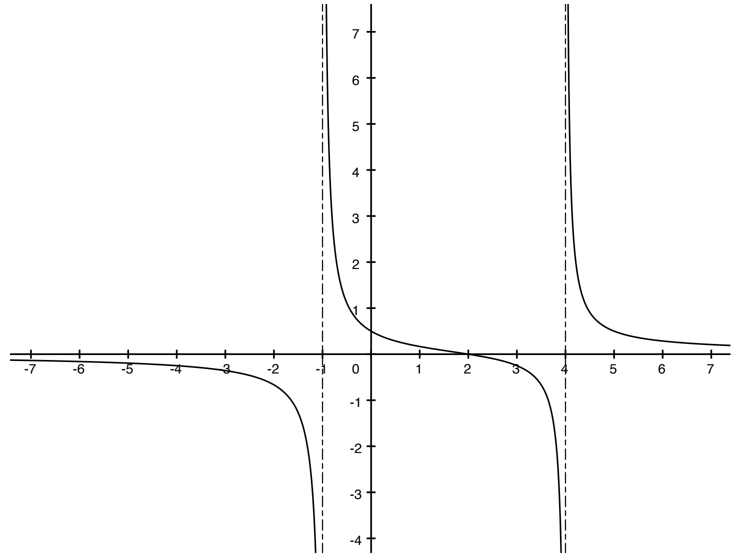
Graph I



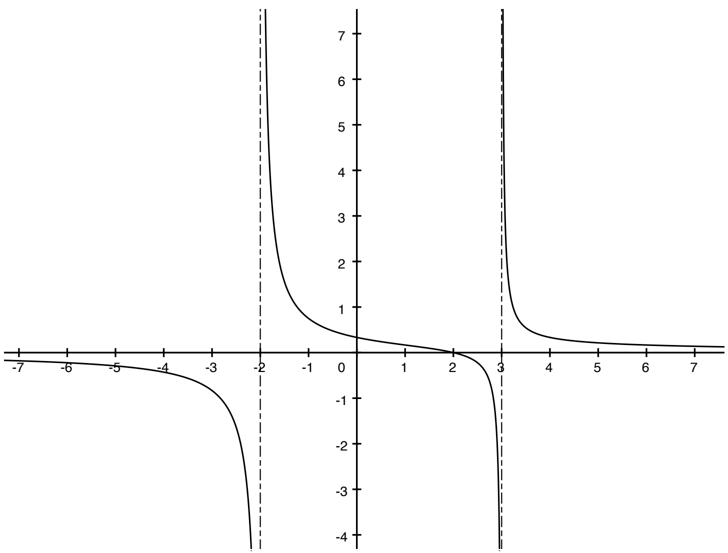
Graph II



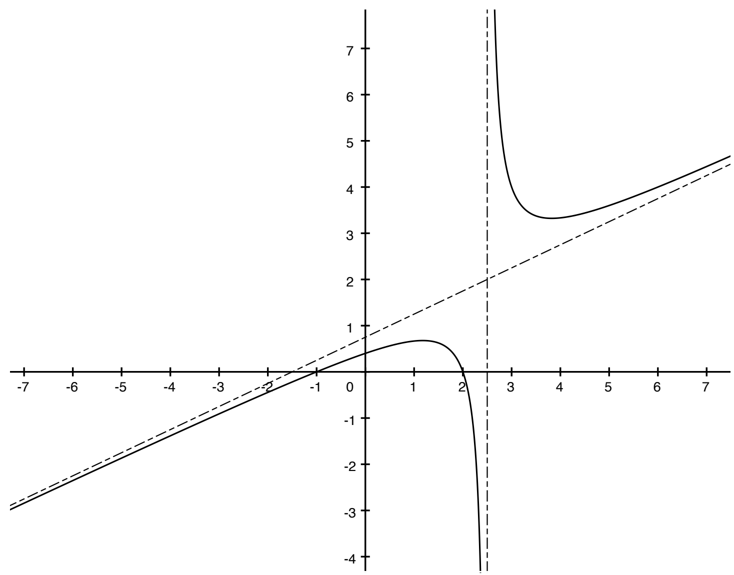
Graph III



Graph IV



Graph V



Graph VI

Solution:

(a) *Graph IV:* $R(x) = \dfrac{x-2}{x^2-3x-4} = \dfrac{x-2}{(x-4)(x+1)}$*, so we are looking for a graph that has one x-intercept,* $x = 2$ *and has vertical asymptotes* $x = -1$ *and* $x = 4$*. Of the given graphs, only Graph IV has these three properties.*

(b) *Graph VI:* $T(x)$ *has a vertical asymptote at* $x = 5/2$*, and only Graph VI has this property.*

(c) *Graph III: The graph of* $T(x) = \dfrac{x^2+x-2}{x^2-2x-3} = \dfrac{(x+2)(x-1)}{(x-3)(x+1)}$ *has x-intercepts* $x = -2$ *and* $x = 1$ *and vertical asymptotes* $x = 3$ *and* $x = -1$*. Only Graph III has these properties.*

(4) (5 pts) Find the inverse function of $f(x) = \dfrac{3x+1}{x+5}$.

Solution: *To find the inverse function, we solve the equation* $y = f(x)$ *for the variable* x *in terms of the variable* y *(and then replace* y *by* x *in the final expression)):*

$$y = \frac{3x+1}{x+5} \implies y(x+5) = 3x+1 \implies yx+5y = 3x+1 \implies yx-3x = 1-5y$$

$$\implies x(y-3) = -5y+1 \implies x = \frac{-5y+1}{y-3} \implies f^{-1}(x) = \frac{-5x+1}{x-3}.$$

(5) The population of Nigeria at the beginning of 2018 was (about) 194 million, and Nigeria's population is growing by (about) 2.7% a year. *Assuming that Nigeria's population continues to grow at this rate...*

Observation: *The assumption that the population is growing by* 2.7% *per year means that the population in year* $t+1$ *will be* 2.7% *bigger than the population in year* t*. I.e., if* $P(t)$ *is the population in year* t*, then*

$$P(t+1) = P(t) + 0.027P(t) = P(t)(1+0.027) = P(t)\cdot 1.027.$$

This implies that

$$P(1) = P(0)\cdot 1.027, \quad P(2) = P(1)\cdot 1.027 = P(0)\cdot(1.027)^2, \quad P(3) = P(2)\cdot 1.027 = P(0)\cdot(1.027)^3$$

and in general

$$P(t) = P(0)(1.027)^t.$$

(a) (2 pts) What will Nigeria's population be in 2030 (to the nearest million)?

Solution: *We use* 2018 *as year* $t=0$*, so that* $P(0) = 194$ *million, and the year* 2030 *corresponds to* $t=12$*, so the population in* 2030 *will be*

$$P(12) = 194\cdot(1.027)^{12} \approx 267 \textit{ million.}$$

(b) (3 pts) In what year will Nigeria's population reach 500 million?

Solution: *If* t_1 *is the number of years from* 2018 *that it takes the population of Nigeria to reach* 500 *million, then*

$$500 = P(t_1) = 194(1.027)^{t_1}.$$

To solve this equation for t_1*, we divide by* 194 *and take natural logs of both sides:*

$$\ln(1.027^{t_1}) = \ln(500/194) \implies t_1\cdot\ln 1.027 = \ln(500/194) \implies t_1 = \frac{\ln(500/194)}{\ln 1.027} \approx 35.536.$$

This means that the population of Nigeria will reach 500 *million about* 35.536 *years after* 2018*, which will be (about mid June) in the year* 2053.