AMS 3 **Midterm Exam 1 – Solutions** Winter 2018

(1) (6 pts) Match each of the given functions to one of the following graphs. (There will be two graphs left over).

(i) $f(x) = -0.5(x-1)^2 + 1$ — Parabola opening down with vertex at $(1,1)$, so graph F.

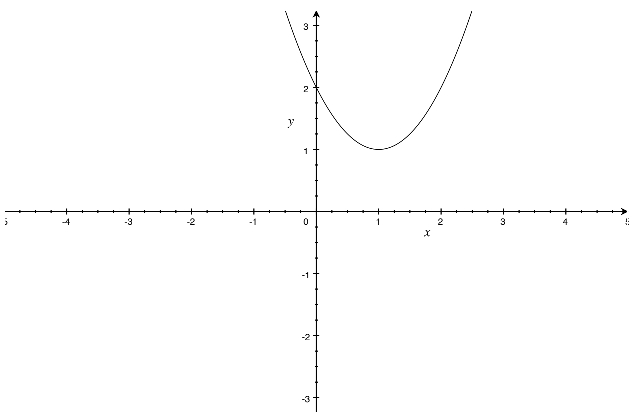
(ii) $g(x) = 0.5(x+1)^2 + 1$ — Parabola opening up with vertex at $(-1,1)$, so graph C.

(iii) $h(x) = \sqrt{2x+2} = \sqrt{2} \cdot \sqrt{x+1}$ — Graph of $y = \sqrt{x}$, shifted to the left by 1 and multiplied by $\sqrt{2}$. In particular $h(0) = \sqrt{2} < 2$, so graph D.

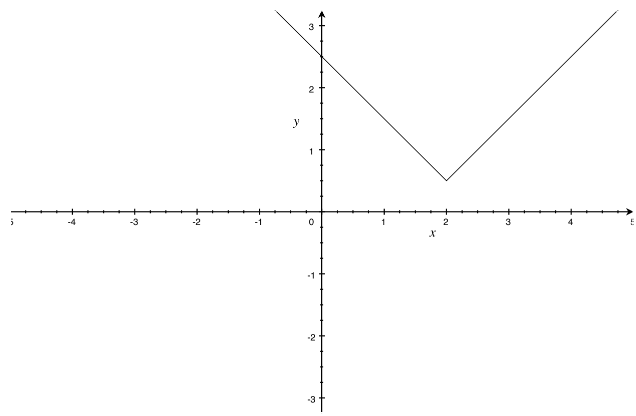
(iv) $k(x) = 2\sqrt{x+1}$ — Graph of $y = \sqrt{x}$, shifted to the left by 1 and multiplied by 2. In particular $k(0) = 2$, so graph G.

(v) $l(x) = |x-2| + 0.5$ — Graph of $y = |x|$, shifted to the right by 2 and up by 0.5, so graph B.

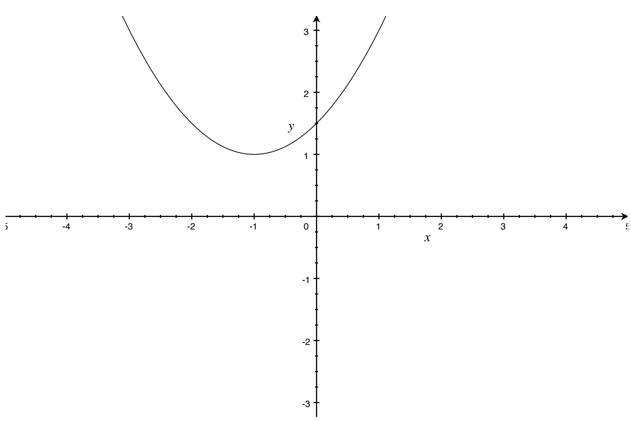
(vi) $p(x) = |2x-4| + 0.5 = 2|x-2| + 0.5$ — Graph of $y = |x|$, shifted to the right by 2, *multiplied by* 2 and shifted up by 0.5. This function goes up more steeply that $l(x)$, so graph H.



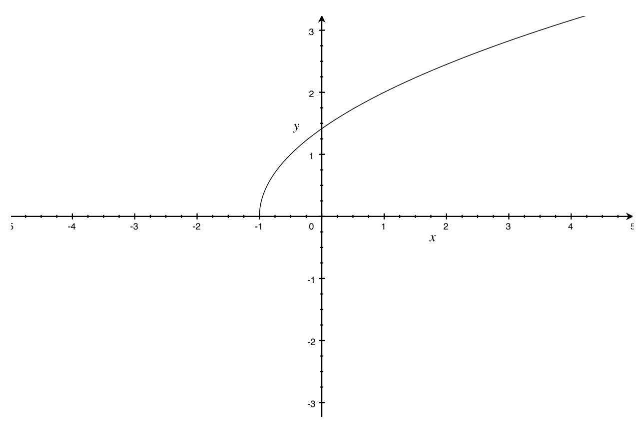
A



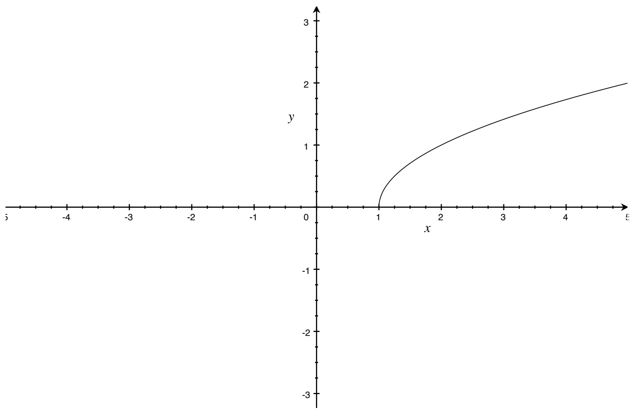
B: $l(x) = |x-2| + 0.5$



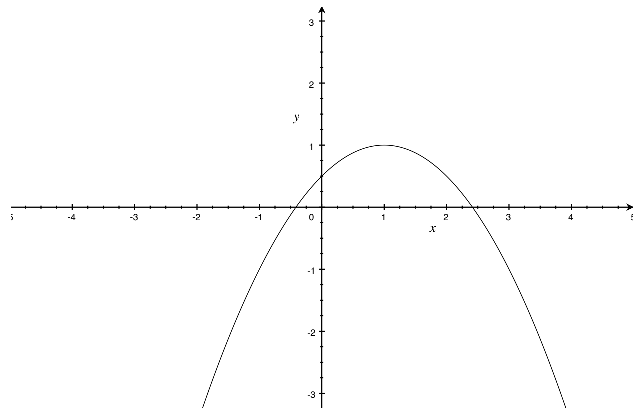
C: $g(x) = 0.5(x+1)^2 + 1$



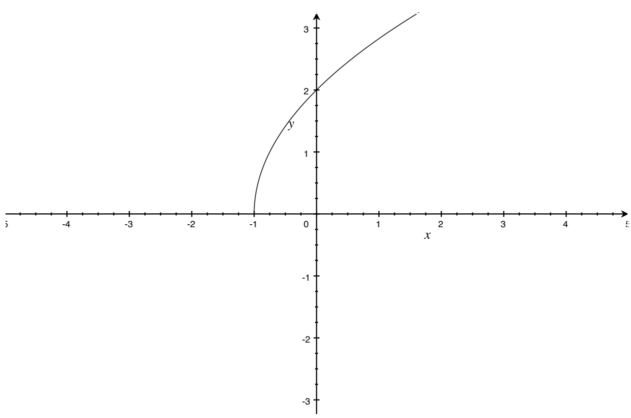
D: $h(x) = \sqrt{2x+2}$



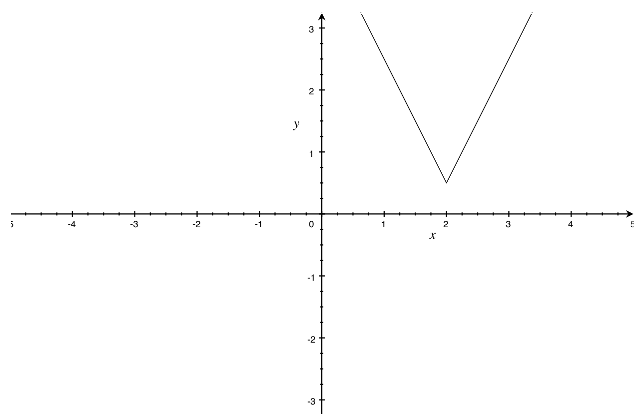
E



F: $f(x) = -0.5(x-1)^2 + 1$

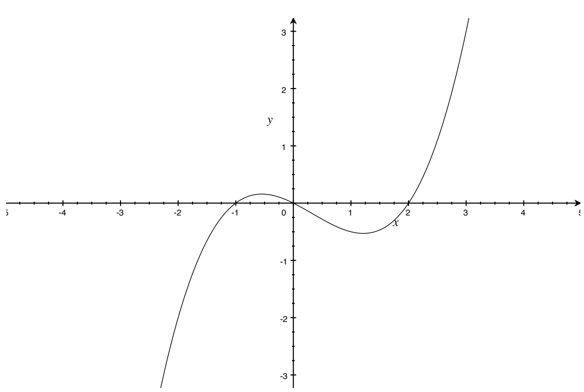


G: $k(x) = 2\sqrt{x+1}$



H: $p(x) = |2x - 4| + 0.5$

(2) (6 pts) The graph of the function $y = f(x)$ is given below. Use this graph to find the x-intercepts of the function $y = f\big(-(x+1)\big)$. Explain your answer (briefly).



$y = f(x)$

The x-intercepts of the graph of $y = f(-(x+1))$ are the points x where $f(-(x+1)) = 0$. Since the x-intercepts of $y = f(x)$ are $x = -1$, $x = 0$ and $x = 2$ (from the graph above), the x-intercepts of $y = f(-(x+1))$ will be the points x such that $-(x+1) = -1$, $-(x+1) = 0$ and $-(x+1) = 2$. These points are

$$-(x+1) = -1 \implies \boxed{x_1 = 0} \quad -(x+1) = 0 \implies \boxed{x_2 = -1} \quad -(x+1) = 2 \implies \boxed{x_3 = -3}$$

(3) (a) (4 pts) The owner of a specialty candle store notices than when he sets the price of a 10-inch spruce-scented candle to be $p_1 = \$12.00$, he sells $q_1 = 46$ of these candles a month, and when he lowers the price to $p_2 = \$10.00$, he sells $q_2 = 50$ candles a month.

Use this data to find a linear model $q = mp + b$ for the weekly demand for 10-inch spruce-scented candles as a function of their price.

The rate of change for *change in quantity/change in price* is

$$m = \frac{50 - 46}{10 - 12} = -2.$$

Using one of the data points and the point-slope formula, we find the equation

$$q - 50 = -2(p - 10) \implies \boxed{q = -2p + 70}$$

which is the linear model we seek.

(b) (4 pts) Calculate the *average rate of change* of the function $f(x) = \sqrt{x+9}$ on the interval $[0, 16]$ and find the equation of the secant line connecting the points $(0, f(0))$ and $(16, f(16))$ on the graph of this function.

First,

$$\text{Average rate of change} = \frac{f(16) - f(0)}{16 - 0} = \frac{5 - 3}{16} = \frac{1}{8}.$$

Second, the secant line connecting these two points has slope equal to the average rate of change, $\frac{1}{8}$, and we find its equation using the point slope formula (and one of the two points — I'm using $(0, 3)$):

$$y - 3 = \frac{1}{8}(x - 0) \implies \boxed{y = \frac{1}{8}x + 3}$$

(4) (a) (4 pts) Find the *minimum* value of the function $f(x) = x^2 + 4x - 5$. Show your work and explain your answer (briefly).

The function $f(x) = x^2 + 4x - 5$ is a quadratic function whose graph is opening up (because the leading coefficient is positive). It's minimum value is the y-value at the *vertex* of the parabola, which you can find in several different ways...

(i) You might remember the formula for the x-coordinate of the parabola:

$$x_v = -\frac{b}{2a} = -\frac{4}{2} = -2,$$

and plug this into the function to find the minimum value (which is the y-coordinate of the vertex):

$$y_v = f(-2) = 4 - 8 - 5 = -9.$$

(ii) You might remember that the x-coordinate of the vertex is halfway between the zeros of the of $f(x)$, which you can find by factoring, for example:

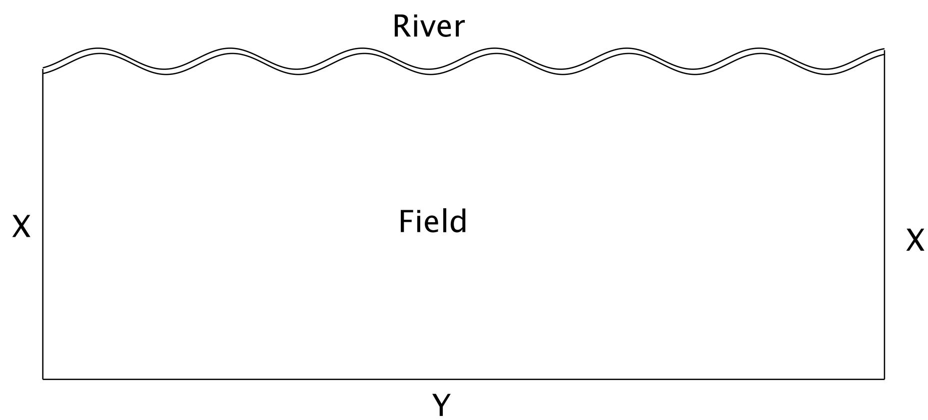
$$f(x) = x^2 + 4x - 5 = (x+5)(x-1),$$

so the zeros are $x_1 = -5$ and $x_2 = 1$, and halfway in between is $x_v = \frac{-5+1}{2} = -2$. Now continue as in (i).
(iii) Complete the square. I.e., write $f(x) = (x-h)^2 + k$, where $h = x_v$ is the x-coordinate of the vertex and $k = y_v$ is the y-coordinate of the vertex:

$$x^2 + 4x - 5 = x^2 + 2 \cdot 2 \cdot x - 5 = \overbrace{x^2 + 2 \cdot 2 \cdot x + 2^2}^{(x+2)^2} - 4 - 5 = (x - (-2))^2 - 9.$$

We have $k = -9$ in this case so this is the minimum value of the function.

(b) (4 pts) A farmer has enough material to put up 300 meters of fence. She wants to enclose a rectangular field that borders a river (see picture), so she only has to fence in three sides. What are the dimensions of the field with the largest area that she can enclose, and what is the largest area?



First, label the dimensions X = width of the field and Y = length of the field. Next, we know that $2X + Y = 300$ (the perimeter of the fenced field = 300, the available fence), and we can use this to express Y in terms of X:

$$Y = 300 - 2X.$$

Finally, we want to *maximize* the area of the field, and we can express this area as a quadratic function of X, using the expression above:

$$A = XY = X(300 - 2X) = -2X^2 + 300X.$$

The graph of this function is a parabola opening downward (because the leading coefficient is $-2 < 0$), so it has a maximum value at its vertex. The X-coordinate of the vertex is halfway between the zeros of the function which are $X_1 = 0$ and $X_2 = 150$ (when $300 - 2X = 0$), so the vertex occurs at $X_v = 75$.
Conclusion: The area is maximized when $X = 75$ and $Y = 300 - 2 \cdot 75 = 150$, and the maximum area is $A = 75 \cdot 150 = 11250$.