Many phenomena exhibit behavior that isn't easily modeled by polynomials. In class we considered several examples, but I will focus on just one here: the relation between a drug's dose and the drug's efficacy.

If none of the drug is taken, then it has no effect, so when the does is 0, the drug's efficacy is also 0. Generally speaking when a very small dose is taken, the drug will have a small positive effect, and as the dose increases, the efficacy also increases.

But only until a certain point, the point of peak efficacy. After that, the negative effects of the drug begin to bring down the efficacy,[†] and as the dose increases beyond the point of peak efficacy, the efficacy decreases back to 0.

Doses beyond this level can be considered toxic and have a negative effect on the patient's health, and therefore can be considered to have negative efficacy. Eventually, the dosage becomes so high as to cause death, which we can symbolically represent by efficacy $-\infty$.

If we graph efficacy as a function of dosage, we might therefore expect to see something like the figure below. In this figure, the points $x = \tau$ (*tau*) and $x = \lambda$ (*lambda*) correspond to the points where the dose becomes toxic and lethal, respectively, and the red line of death can be thought of as a vertical asymptote to the graph.

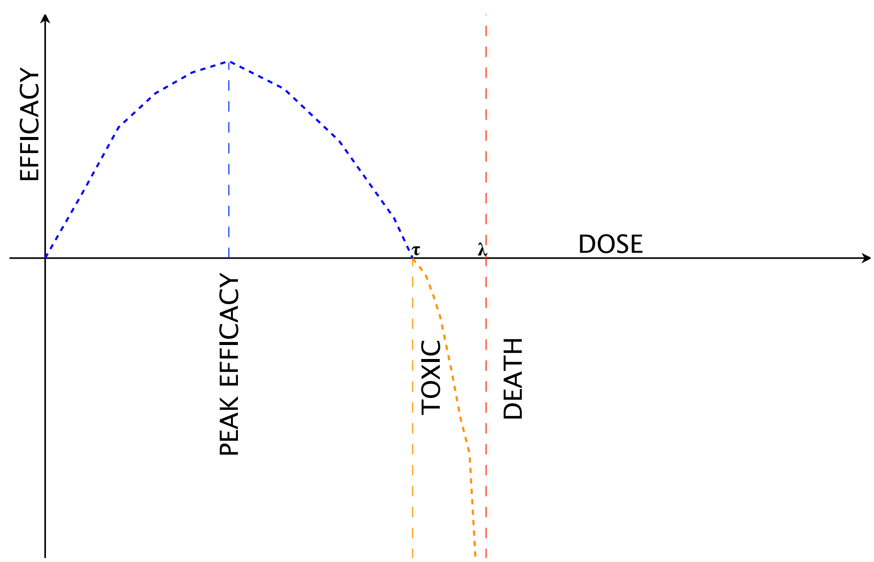

Figure 1: Drug efficacy as a function of dosage.

Such a relation cannot be modeled by a polynomial—because polynomials don't have vertical asymptotes—which leads us to rational functions.

The rest of this note is purely mathematical. The goal being to find a relatively simple rational function that produces a graph that looks like the one above. But before we proceed, I want to reemphasize the point that I made in class: *finding a function whose graph has the characteristics we seek is the (relatively) easy part of this type of problem. There are generally infinitely many different ways to do this. The real challenge is finding a function $F(x)$ and*

[†]These may include side effects or the stress on a body's organs caused by the processing of the chemicals in the drug, for example.

a scientific explanation for why the relation $y = F(x)$ describes how efficacy (y) is related to dosage (x).

Back to the mathematical problem of finding a rational function whose graph matches the one in Figure 1. I'll do this in three stages. First I'll find a function whose graph matches the blue part of the graph, then I'll adjust that function to give it a vertical asymptote and finally I will adjust it a little more, to make it pretty.

(1) *Matching the blue part*: This part of the graph is fairly symmetric around its maximum. It looks like the top of a parabola with x-intercepts at $x = 0$ and $x = \tau$. To make things more concrete, let's suppose that $\tau = 2$. This means that to match the blue part of the graph, we need a quadratic function $p(x)$ that has zeros at $x = 0$ and $x = 2$, whose graph opens *down*. This means that $p(x) = ax(2-x)$, where a is a positive constant. The vertex of $p(x)$ occurs at $x = 1$ (why?), so the y-coordinate of the vertex is $p(1) = a \cdot 1 \cdot (2-1) = a$. If the coordinate axes are on the same scale, it would appear that $a \approx 1.25$ or so. Which means that $p(x) = 1.25x(2-x)$. whose graph is displayed below (in black).

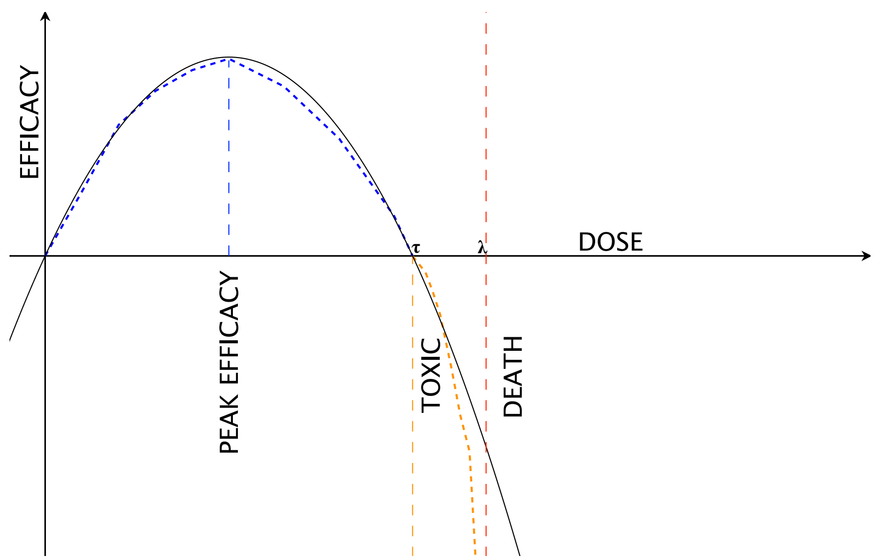

Figure 2: Graph of $y = 1.25x(2-x)$, superimposed on dose-efficacy graph.

It does a pretty good job, with one obvious and expected flaw: it doesn't have a vertical asymptote. To fix this, we form a rational function $F(x) = p(x)/q(x)$, such that $q(x) > 0$ for $0 \leq x < \lambda$ and $g(\lambda) = 0$, where $x = \lambda$ is the vertical asymptote (corresponding to a lethal dose). The simplest function that does this is $g(x) = \lambda - x$, and to make a concrete choice, let's say that $\lambda = 2.4$. I.e.,

$$F(x) = \frac{1.25x(2-x)}{2.4-x}.$$

The graph of this function appears in Figure 3.

It has the right zeros and the right vertical asymptote, but it has lost the symmetry and the height of its peak. The height can be fixed by scaling (i.e., multiplying by a large enough constant), so first I'll fix the lack of symmetry.

To regain the symmetry, we change the denominator to make it also symmetric around the point $x = 1$. The point 2.4 is 1.4 units above 1, and the point $x = -0.4$ is 1.4 units below $x = 1$,

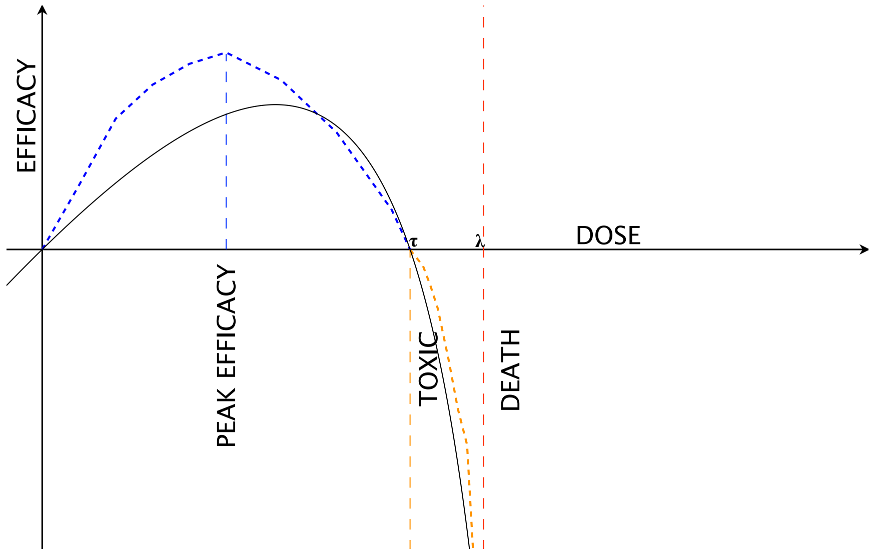

Figure 3: Graph of $y = \dfrac{1.25x(2-x)}{2.4-x}$, superimposed on dose-efficacy graph.

so the function $r(x) = (2.4 - x)(x + 0.4)$ is symmetric around $x = 1$ (as you should check) and therefore the rational function

$$H(x) = \frac{1.25x(2-x)}{(2.4-x)(x+0.4)}$$

is also symmetric around $x = 1$. Additionally, $H(x)$ retains the vertical asymptote at $x = 2.4$ (and adds another one at $x = -0.4$). Its graph is displayed in Figure 4, with the portion for $x < 0$ omitted.

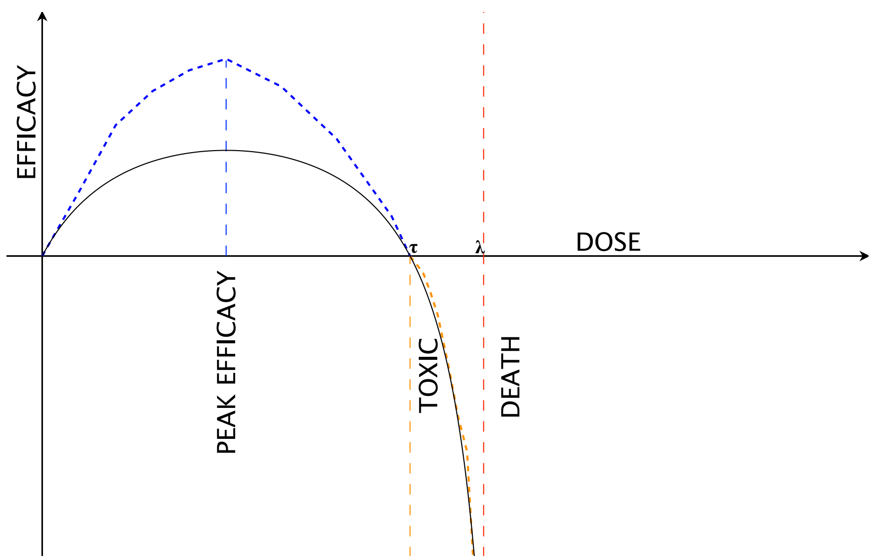

Figure 4: Graph of $H(x) = \dfrac{1.25x(2-x)}{(2.4-x)(x+0.4)}$.

Not surprisingly, the graph of $H(x)$ is not high enough, and experimenting a bit, we find that the the function $D(x) = 2.1H(x)$ has the right height, is symmetric around the peak and has the requisite vertical asymptote at $x = \lambda$.

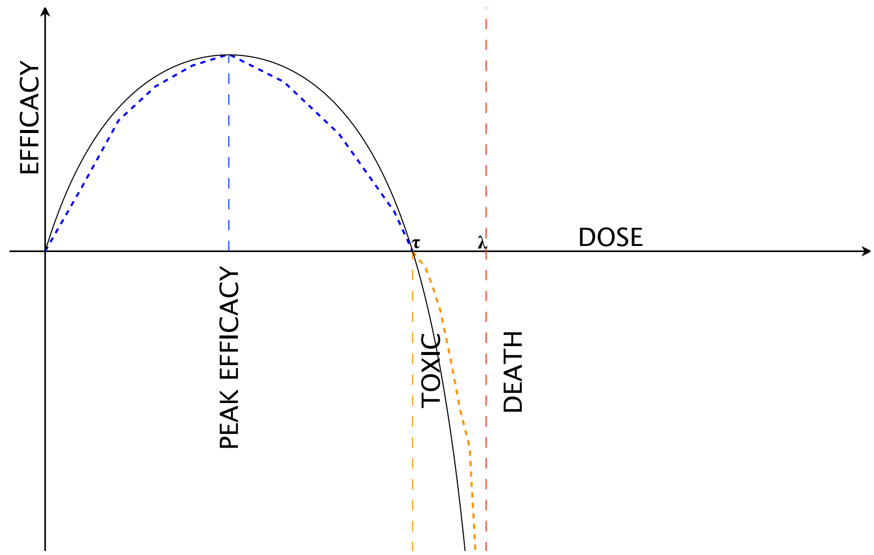

Figure 5: Graph of $D(x) = 2.1 \cdot \dfrac{1.25x(2-x)}{(2.4-x)(x+0.4)}$.

And there you go.