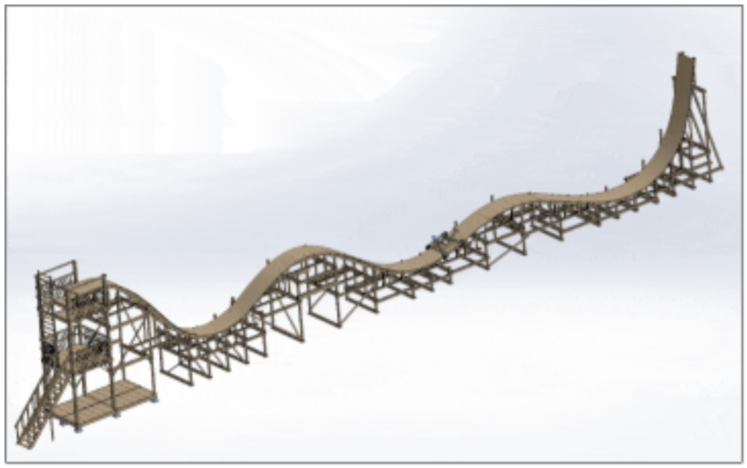
In this example, we want to find a polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$ whose graph looks something like the roller coaster in the image below.† I.e., we want a function $f(x)$ that models the height $y = f(x)$ at distance x from the starting point.



The roller coaster in the picture has 5 turning points after it starts (though eventually, we will also require that the starting point of the roller coaster is a turning point) and to make our work a little easier, we will look for a graph with only three turning points (after it starts).

The ground in the image above can be thought of as the x-axis, so the roller coaster graph (henceforth the RC graph) lies above the x-axis, at least over the interval that the picture covers. We can also assume that the beginning of the RC graph is on the y-axis ($x = 0$). This means that we are looking for a graph that looks something like the one in Figure 1.

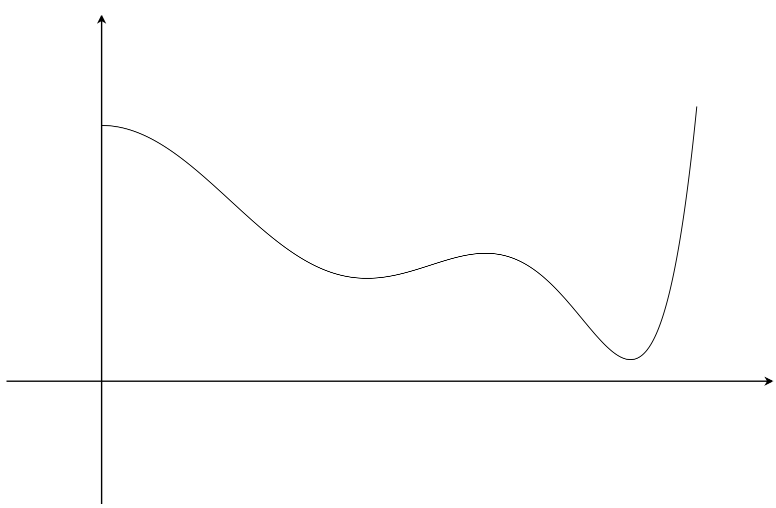


Figure 1: RC graph

†Inspired by the annual MIT roller coaster project — google it.

It is generally easier to find polynomials if we know where their zeros are, so we translate the RC graph down, so the second turning point is a zero, as in Figure 2.

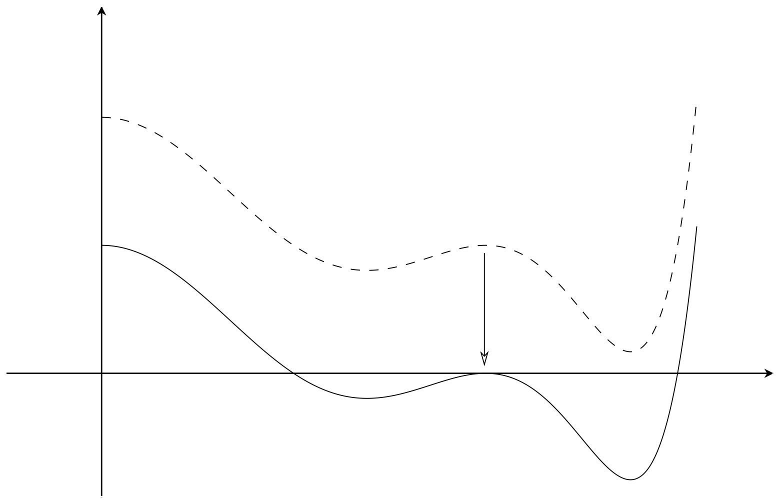


Figure 2: RC graph — shifted down

Finally, we put in scales on the axes because we need to have numbers if we are going to find an explicit polynomial. In particular, we can choose where the height of the starting point and the positions of the zeros to make our work as simple as possible. This is done in the next figure.

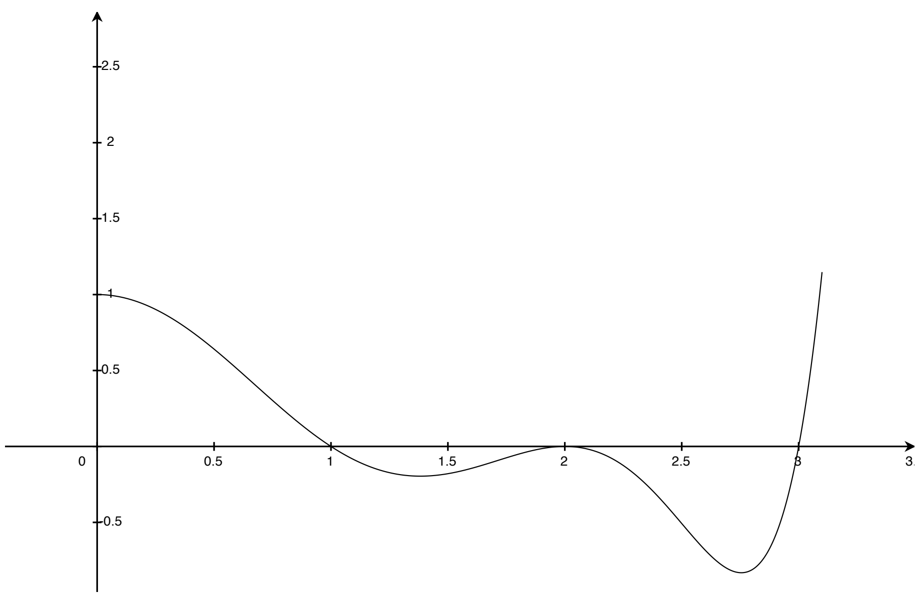


Figure 3: RC graph, shifted with scales on the axes

A polynomial function whose graph looks like the one above has zeros at $x = 1$, $x = 2$ and $x = 3$. Moreover, the zeros $x = 1$ and $x = 3$ have *odd* multiplicity (why?) and $x = 2$ is a zero of *even* multiplicity (why?). Since one always begins with the simplest model possible, we assume that $x = 1$ and $x = 3$ have multiplicity one and $x = 2$ has multiplicity two, and our first guess is the polynomial

$$f_1(x) = a(x-1)(x-2)^2(x-3),$$

where a is chosen to make $f(0) = 1$ to match the graph in Figure 3. This means that

$$1 = f_1(0) = a(0-1)(0-2)^2(0-3) = 12a,$$

so $a = 1/12$ and

$$f_1(x) = \frac{1}{12}(x-1)(x-2)^2(x-3) = \frac{1}{12}x^4 - \frac{2}{3}x^3 + \frac{23}{12}x^2 - \frac{7}{3}x + 1.$$

The graph of this function appears in Figure 4 below (solid line) together with our desired RC graph (broken line). As you can see, $f_1(x)$ is not a good match for the RC graph.

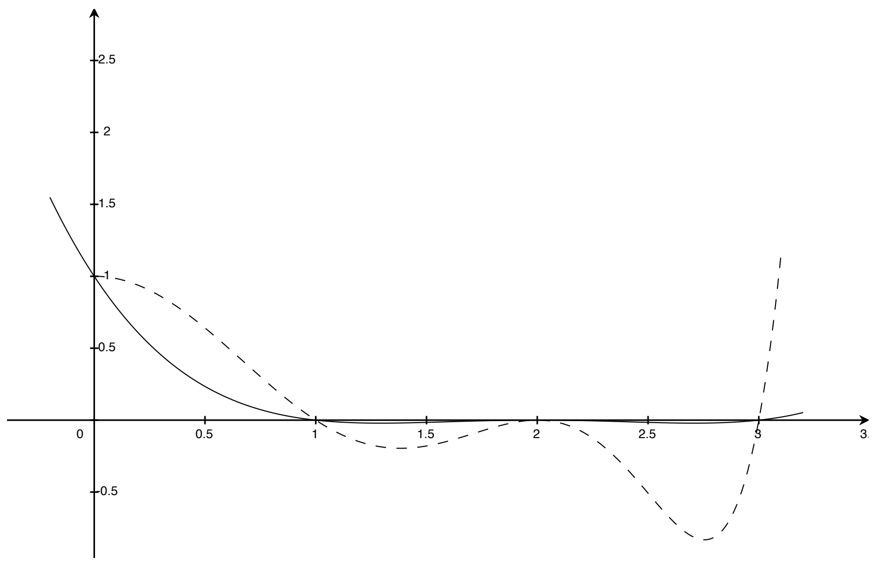


Figure 4: RC Graph and graph of $f_1(x) = \frac{1}{12}x^4 - \frac{2}{3}x^3 + \frac{23}{12}x^2 - \frac{7}{3}x + 1$

It does have the right zeros with the right multiplicities and the right value at $x = 0$, which is not surprising because this is how we built it. Ignoring for the moment the fact that the graph $y = f_1(x)$ more or less hugs the x-axis between $x = 1$ and $x = 3$, the real problem is that the behavior at $x = 0$ is wrong. We want the RC graph to flatten out (go horizontal) at $x = 0$, which it doesn't for $f_1(x)$. One way to achieve this is to require that $(0, 1)$ be a turning point of the RC graph. On the other hand, we want to do this in a way that preserves all the other features (where the zeros are and their multiplicities, for example).

It turns out that there is a relatively easy way to make this happen, based on the following two facts.

(a) *If $G(x)$ is an* ***even*** *function, i.e., $G(-x) = G(x)$ for all x, then the point $(0, G(0))$ is a turning point on the graph of $G(x)$.*

This is because the graph of an even function is *symmetric around the y-axis*, so if the function G is decreasing to the right of $x = 0$, then it must be increasing to the left of $x = 0$, and *vice versa*.

(b) *Given any function $g(x)$, the function $G(x) = g(x) \cdot g(-x)$ is even.*

This follows from the fact that

$$G(-x) = g(-x) \cdot g(-(-x)) = g(-x)g(x) = g(x)g(-x) = G(x).$$

It is true that $g(x) + g(-x)$ will also be an even function. The advantage of taking the product $g(x)g(-x)$, is that it *preserves the zeros* of $g(x)$ (adding more, in many cases). This is important to us in this case at hand.

Forming the function $F(x) = f_1(x)f_1(-x)$, we find that

$$\begin{aligned} F(x) &= \left(\frac{1}{12}(x-1)(x-2)^2(x-3)\right)\left(\frac{1}{12}(-x-1)(-x-2)^2(-x-3)\right) \\ &= \frac{1}{144}(x+3)(x+2)^2(x+1)(x-1)(x-2)^2(x-3) \\ &= \frac{1}{144}(x^2-1)(x^2-4)^2(x^2-9) \end{aligned}$$

This is a polynomial of degree 8 with zeros at $x = -3, -2, -1, 1, 2$ and 3 (with multiplicity 1 at $x = \pm 1, \pm 3$ and multiplicity 2 at $x = \pm 2$). It's graph appears in Figure 5 below, where the portion to the right of $x = 0$ is highlighted in blue, and is a perfect match for the *shifted* RC graph.

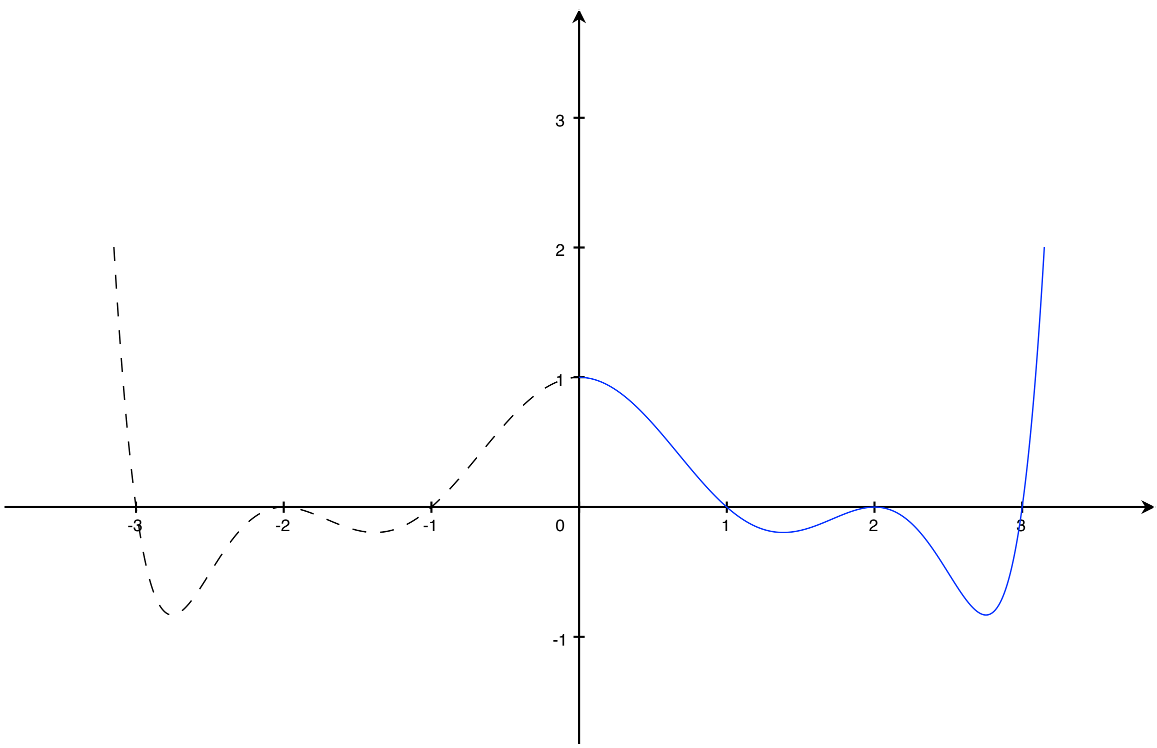


Figure 5: Graph of $F(x) = \frac{1}{144}(x^2-1)(x^2-4)^2(x^2-9)$

This means that the roller coaster can be modeled by a polynomial of the form

$$RC(x) = F(x) + c = \frac{1}{144}(x^2-1)(x^2-4)^2(x^2-9) + c$$

where $c > 0$ must be big enough to shift the entire graph above the x-axis, for example $c = 1$ works. The graph of the function $RC(x) = \frac{1}{144}(x^2 - 1)(x^2 - 4)^2(x^2 - 9) + 1$ is pictured below in Figure 6.

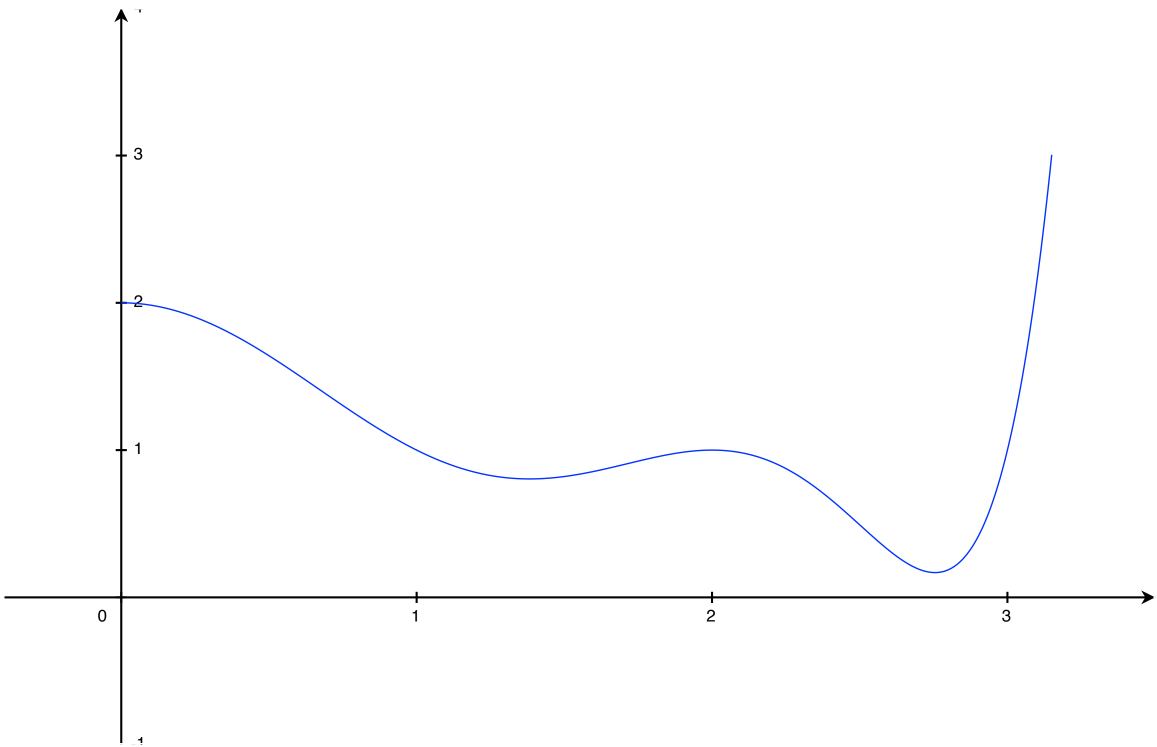


Figure 6: Graph of $RC(x) = \frac{1}{144}(x^2 - 1)(x^2 - 4)^2(x^2 - 9) + 1$

Eureka!