1. Find the polynomial of smallest degree whose graph matches the one in Figure 2, below.

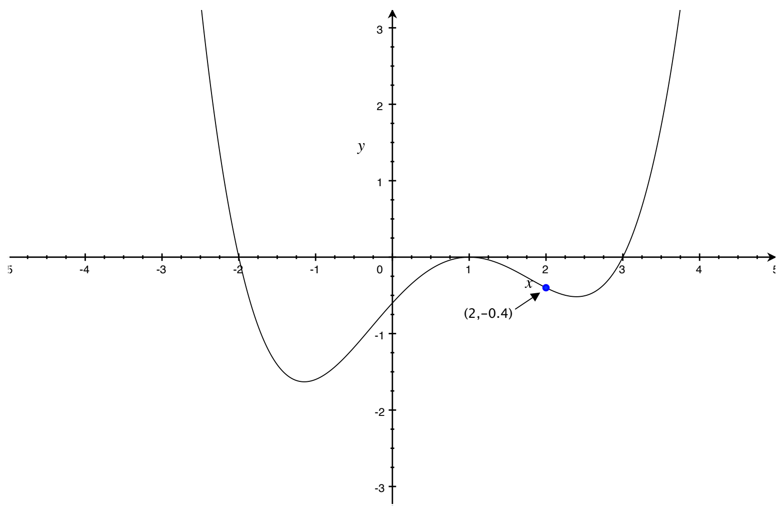


Figure 1: Graph for problem 2.

Solution: *This graph has x intercepts -2, 1 and 3, so the polynomial we are looking for has factors $(x+2)$, $(x-1)$ and $(x-3)$. The graph crosses the x axis at -2 and 3, so the factors $(x+2)$ and $(x-3)$ appear with odd powers, and the graph touches the x axis at 1, so the factor $(x-1)$ appears with an even power. Since we want the polynomial to have the smallest degree possible, we choose the smallest powers possible for the three factors and conclude that*

$$p(x) = a(x+2)(x-1)^2(x-3),$$

for some constant a. To find a we use the point $(2,-0.4)$ on the graph. We have

$$-0.4 = p(2) = a(2+2)(2-1)^2(2-3) = a\cdot(-4) \implies a = \frac{-0.4}{-4} = 0.1,$$

so

$$p(x) = 0.1(x+2)(x-1)^2(x-3).$$

2. Follow steps 1 through 5 on page 205 of the textbook to analyze the polynomial

$$f(x) = 0.5(x+1)(x-2)^3.$$

Solution:

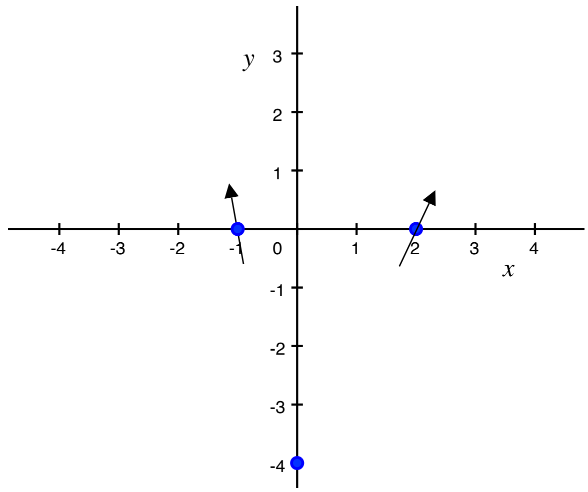
(a) *End behavior: $f(x) = 0.5(x+1)(x-2)^3 = 0.5x^4 - 2.5x^3 + 3x^2 + 2x - 4$ has even degree and a positive leading coefficient, so $f(x) \to +\infty$ on both ends.*

(b) *Intercepts: The x intercepts are $x = -1$ and $x = 2$ and the y intercept is -4.*

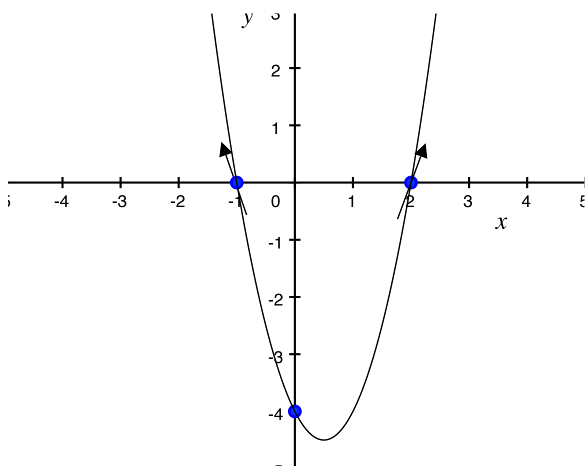
(c) *Zeros: The zeros of $f(x)$ are $x_1 = -1$ (multiplicity 1) and $x_2 = 2$ (multiplicity 3). Since the multiplicities of both zeros are odd, the graph crosses the x axis at both zeros.*

(d) *The graph has at most $4 - 1 = 3$ turning points.*

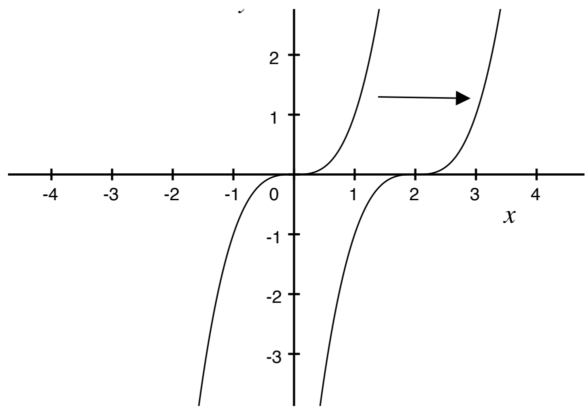
(e) *Graph: Using the points we have, the end behavior and the behavior (crossing) at the zeros, we can draw an initial, partial sketch:*



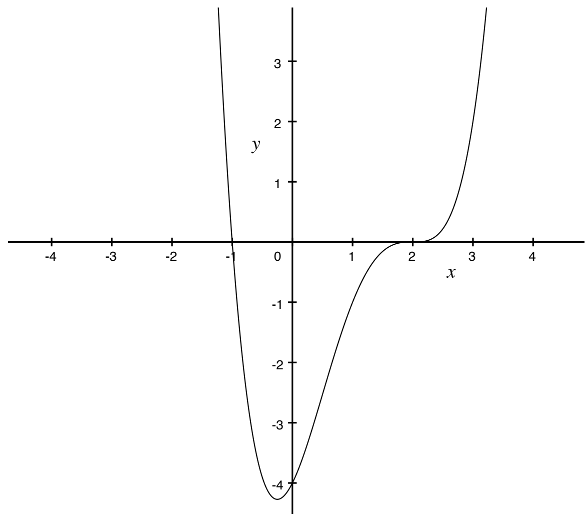
If you simply connect the dots you will get something that looks like a parabola:



However this isn't quite right, because it misses the behavior around the point $x = 2$. The factor $(x - 2)$ appears with multiplicity 3, and because of this, the graph of $f(x)$ around the point $(2, 0)$ will look like the graph of $y = x^3$ around $(0, 0)$:



Combining the look of the graph around $x = 2$ with the earlier information, we get the graph:



3. Find the *rational* zeros of the polynomial

$$p(x) = 6x^3 - 17x^2 - 5x + 6.$$

Solution: *Recall that if k/l is a rational zero of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with* integer *coefficients, then k must be factor of the constant coefficient a_0 and l must be a factor of the leading coefficient a_n.*

This means that if k/l is a zero of $p(x)$, above, then k and l must both be factors of 6, i.e., $k = \pm 1, \pm 2, \pm 3$ or ± 6 and $l = 1, 2, 3$ or 6 (we only need to $\pm$ on the numerator). This means that we need to check the rational numbers,

$$\frac{k}{l} = \pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{1}{6},\ \pm\frac{3}{2}\ \text{and}\ \pm\frac{2}{3}.$$

(Can you see why there are no others?)

Checking: $p(1) = -10 \neq 0$, $p(-1) = -12 \neq 0$, $p(2) = -24 \neq 0$, $p(-2) = -100 \neq 0$, $p(3) = 0$*! Now, to save time (before checking the other* 13 *candidates) we divide* $p(x)$ *by the factor* $(x-3)$*:*

$$\begin{array}{r|l} & 6x^2 \quad + x - 2 \\ \hline x-3 & 6x^3 - 17x^2 - 5x + 6 \\ & -6x^3 + 18x^2 \\ \hline & \qquad x^2 - 5x \\ & \qquad -x^2 + 3x \\ \hline & \qquad\qquad -2x + 6 \\ & \qquad\qquad 2x - 6 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

This means that $6x^3 - 17x^2 - 5x + 6 = (x-3)(6x^2 + x - 2)$*, and any remaining zeros of* $p(x)$ *will also be zeros of* $p_1(x) = 6x^2 + x - 2$*. At this point, we can repeat the process above for* $p_1(x)$ *— find the (shorter) list of candidates and check — or we can save time by using the quadratic formula and seeing if the zeros of* $p_1(x)$ *are rational:*

$$\frac{-1 \pm \sqrt{1 + 4 \cdot 2 \cdot 6}}{12} = \frac{1 \pm 5}{12} = \frac{1}{2} \quad \text{or} \quad -\frac{1}{3}.$$

Conclusion: the rational zeros of $p(x)$ *are* 3, 1/2 *and* $-1/3$.

4. Consider the rational function:

$$R(x) = \frac{x^2 - x - 2}{2x^2 + 5x - 3}.$$

(a) Factor the numerator and denominator of $R(x)$. What is the domain of this function?

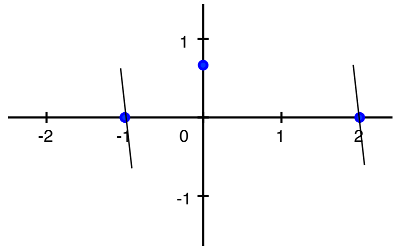
Solution:

$$x^2 - x - 2 = (x-2)(x+1) \qquad \text{and} \qquad 2x^2 + 5x - 3 = (2x-1)(x+3).$$

The domain of $R(x)$ *is* $\{x | x \neq -3 \text{ and } x \neq 1/2\}$.

(b) Find the x and y intercepts of $R(x)$ and determine the behavior of the function at each x intercept ('crossing' or 'touching').

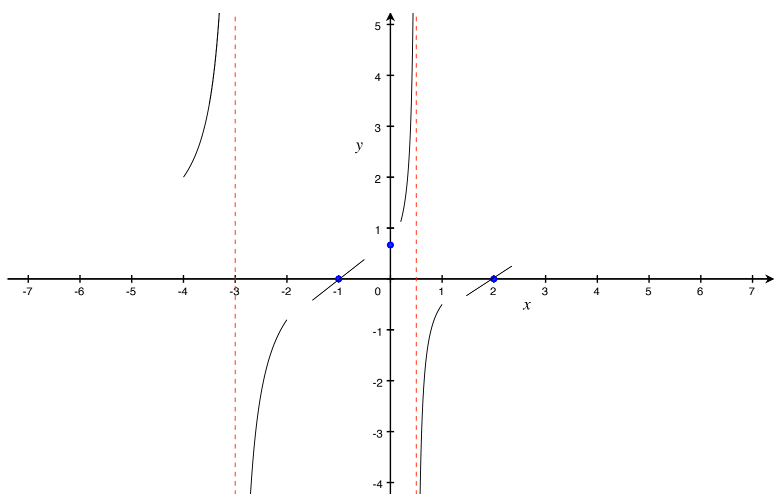
Solution: *The* x *intercepts are the zeros of the numerator,* $x_1 = -1$ *and* $x_2 = 2$*, and the* y *intercept is* $R(0) = 2/3$*. Since the factors* $(x+1)$ *and* $(x-2)$ *both appear to an odd power in the numerator, the graph* $y = R(x)$ *crosses the* x *axis at these points. So far, our graph looks something like the figure below, though the directions the graph is crossing the* x *axis at the intercepts might be wrong.*



(c) Find the vertical asymptotes of $R(x)$ and determine the behavior of $R(x)$ on either side of each vertical asymptote.

Solution: *The vertical asymptotes occur at the zeros of the denominator, so they are the lines $x = -3$ and $x = 1/2$. The sign of $+/-$ of $R(x)$ can only change at zeros or at the vertical asymptotes, and since $R(0) = 2/3 > 0$, it follows that $R(x) > 0$ to the left of the asymptote $x = 1/2$, so $R(x) \to \infty$ on the left side of the asymptote. Similarly, since $R(1) = -1/2 < 0$, it follows that $R(x) \to -\infty$ on the right side of $x = 1/2$.*

For the asymptote $x = -3$, we have $R(-2) = -4/5 < 0$ so $R(x) \to -\infty$ to the right of $x = -3$, and since $R(-4) = 2 > 0$ it follows that $R(x) \to \infty$ to the left of $x = -3$. Including this information we can improve the sketch of the graph to the one below:



(notice how I changed the direction of the crossings at $x = -1$ and $x = 2$.)

(d) Find the horizontal or oblique asymptote of $R(x)$, or say why no such line exists. If there is a horizontal/oblique asymptote, find any point(s) of intersection of the graph of $R(x)$ and the asymptote.

Solution: Since the degrees of the numerator and denominator of $R(x)$ are equal, and the ratio of their leading coefficients is $\frac{1}{2}$, the line $y = \frac{1}{2}$ is a horizontal asymptote to the graph. The graph of $R(x)$ and this asymptote intersect at the point where $R(x) = \frac{1}{2}$:

$$\frac{x^2 - x - 2}{2x^2 + 5x - 3} = \frac{1}{2} \implies \cancel{2x^2} - 2x - 4 = \cancel{2x^2} + 5x - 3 \implies -7x = 1 \implies x = -\frac{1}{7}$$

(e) Find the intervals where $R(x) > 0$ and $R(x) < 0$.

Solution: *We did most of the work for this back in part (c), but I will summarize the results again here using the fact that the sign (+/−) of this function can only change at a zero or at vertical asymptote. This means that the sign of* $R(x)$ *is constant on the intervals:*

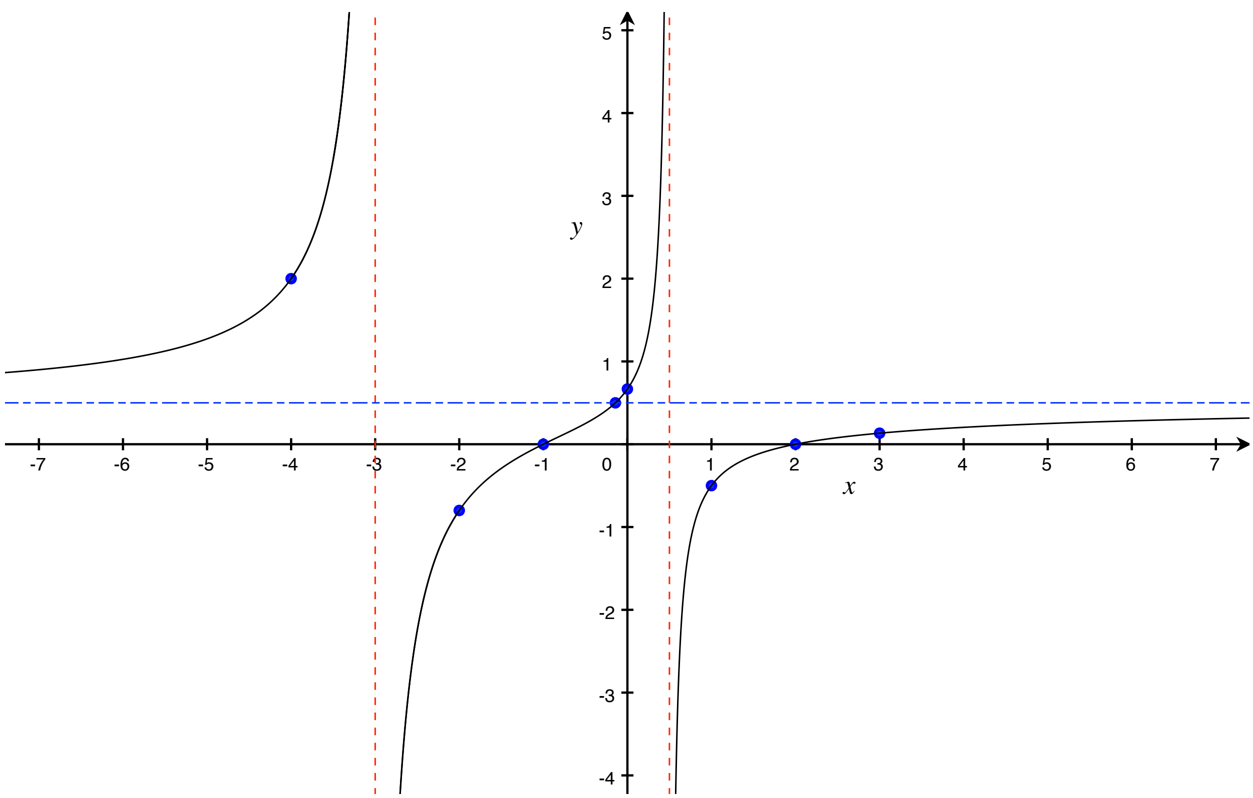
$$(-\infty,-3),\ (-3,-1),\ (-1,1/2),\ (1/2,2)\ \textit{and}\ (2,\infty).$$

We sample a point in each interval, and the sign of the function at that point is the sign of the function in the interval.

interval	$(-\infty,-3)$	$(-3,-1)$	$(-1,1/2)$	$(1/2,2)$	$(2,\infty)$
point x	-4	-2	0	1	3
$R(x)$	2	$-4/5$	$2/3$	$-1/2$	$2/15$
Sign	$+$	$-$	$+$	$-$	$+$
point on graph	$(-4,2)$	$(-2,-4/5)$	$(0,2/3)$	$(1,-1/2)$	$(3,2/15)$

Additionally, we can use the extra points to help plot the graph.

(f) Use all the information you have found to sketch the graph of $R(x)$.



5. For the functions

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = x^2 - 1$$

find the composite functions

$$f \circ f, \quad f \circ g, \quad g \circ f \quad \text{and} \quad g \circ g$$

and find the domain of each one.

Solution:

- $f \circ f = \dfrac{1}{\frac{1}{x+2} + 2} = \dfrac{1}{\frac{1}{x+2} + \frac{2(x+2)}{x+2}} = \dfrac{1}{\frac{2x+5}{x+2}} = \dfrac{x+2}{2x+5}$ *Domain:* $\{x | x \neq -2, -5/2\}$.
- $f \circ g = \dfrac{1}{(x^2-1)+2} = \dfrac{1}{x^2+1}$ *Domain: all* x.
- $g \circ f = \left(\dfrac{1}{x+2}\right)^2 - 1 = \dfrac{1-(x+2)^2}{(x+2)^2}$ *Domain:* $\{x | x \neq -2\}$.
- $g \circ g = (x^2-1)^2 - 1 = x^4 - 2x^2 - 2$ *Domain: all* x.

6. Find the inverse functions of $H(x) = \dfrac{x+1}{2x+3}$ and $G(x) = 4x^3 - 1$. Find the domains of both inverses.

Solution: *To find the inverse of a function* $f(x)$*, we solve* $y = f(x)$ *for* x *in terms of* y*, and then swap the* x *and* y*.*

$$H(x) : y = \frac{x+1}{2x+3} \implies y(2x+3) = x+1 \implies 2yx + 3y = x + 1 \implies 2yx - x = 1 - 3y$$

$$\implies x(2y-1) = 1 - 3y \implies x = \frac{1-3y}{2y-1} \implies H^{-1}(x) = \frac{-3x+1}{2x-1}.$$

The domain of $H^{-1}(x)$ *is* $\{x | x \neq 1/2\}$.

$$G(x) : y = 4x^3 - 1 \implies 4x^3 = y + 1 \implies x^3 = \frac{y+1}{4} \implies x = \sqrt[3]{\frac{y+1}{4}} \implies G^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$$

The domain of $G^{-1}(x)$ *is all* x.

7. The surface area of an inflatable globe of radius r is

$$S = 4\pi r^2.$$

The radius of the ball increases as a function of time t (in seconds), according to the rule $r = 2t^{1/3}$. Find the surface area of the ball as a function of t.

Solution: *To express the surface area as a function of time, we need to find the composite function* $S \circ r$*:*

$$S \circ r = 4\pi(2t^{1/3})^2 = 16\pi t^{2/3}.$$

8. According to U.S. Census bureau, the world's (human) population in 2013 was 7.13 billion and was growing at a rate of 1.1% per year. E.g., in 2014, the population was 1.1% greater than in 2013, so it was

$$7.13 + (1.1\%) \cdot 7.13 = 7.13 \cdot 1.011 \approx 7.208 \text{ billion.}$$

(a) Assuming that the world maintains this rate of growth, find the function

$$P(t) = \text{ world's population, in billions, } t \text{ years after 2013.}$$

Solution: $P(t) = 7.13(1.011)^t$.

(b) According to the model you found in (a), what will the world's population be in 2050? Round your answer to the nearest million.

Solution: *The year* 2050 *is* 37 *years after* 2013*, so the population in* 2050 *will be*

$$P(37) = 7.13(1.011)^{37} = 10.68765\ldots \textit{ billion.}$$

Since we want to round to the nearest million, we round the answer to three decimal places. I.e., the population in 2050 *will be about* 10.688 *billion or* 10 *billion,* 688 *million.*

(c) According to this model, in what year will the world's population reach 20 billion?

Solution: *We want to find the value of* t *such that* $P(t) = 20$*...*

$$7.13(1.011)^t = 20 \implies (1.011)^t = \frac{20}{7.13} \implies \ln((1.011)^t) = \ln\left(\frac{20}{7.13}\right)$$

$$\implies t\ln(1.011) = \ln\left(\frac{20}{7.13}\right) \implies t = \frac{\ln(20/7.13)}{\ln(1.011)} \approx 94.28$$

I.e., the population will reach 20 *billion a little more than* 94 *years after* 2013*, so in the year* 2107*.*

9. Newton's law of cooling/heating (see section 4.8) states that the temperature $u(t)$ at time t of a body immersed in a medium of constant ambient temperature T, can be modeled by the function

$$u(t) = T + (u_0 - T)e^{kt},$$

where u_0 is the initial temperature of the body (at the time of immersion) and $k < 0$ is a constant related to the heat conduction properties of the body.

A metal ball is heated to 500° Celsius and then immersed in a vat of ice water, that is kept at 0° Celsius. After 10 minutes the temperature of the ball is 400° Celsius. When will the ball reach a temperature of 100° Celsius?

Solution: *The initial temperature of the ball is* $u_0 = 500$ *and the ambient temperature is* $T = 0$*, so according to Newton's law, the temperature of the ball at time* t *(measured in minutes here) is*

$$u(t) = 0 + (500 - 0)e^{kt} = 500e^{kt}.$$

To find the value of k*, we use the second data point:* $u(10) = 400$*:*

$$400 = u(10) = 500e^{k \cdot 10} \implies e^{10k} = \frac{400}{500} \implies 10k = \ln 0.8 \implies k = \frac{1}{10}\ln 0.8 \; (\approx -0.0223),$$

so that

$$u(t) = 500e^{(\frac{1}{10} \cdot \ln 0.8)t}.$$

To find the time when the temperature will be 100*, we solve the equation* $u(t) = 100$ *for* t*:*

$$500e^{(\frac{1}{10} \cdot \ln 0.8)t} = 100 \implies e^{(\frac{1}{10} \cdot \ln 0.8)t} = \frac{100}{500} \implies (\frac{1}{10} \cdot \ln 0.8)t = \ln 0.2 \implies t = \frac{10 \ln 0.2}{\ln 0.8}$$

Conclusion: The ball will reach a temperature of 100° *Celsius after* $t \approx 72.126$ *minutes.*

Comment: *I didn't use the rounded (truncated) version of* k *in the last calculation — I only rounded my final answer. If we use* $k \approx -0.0223$ *in the calculation above, we would find that the ball reached* 100 *degrees after* $t \approx 17.172$ *minutes. This is a difference of about* 2.76 *seconds. Not a big difference, but some cases, we might want the extra precision.*

10. A video spreads virally on a certain social media platform according to the model

$$N(t) = \frac{1,000,000}{1 + be^{-kt}},$$

(the logistic model), where $N(t)$ is the number of people who have seen the video, and the parameters b and k are (currently) unknown. At 1:00 am on Monday, 1000 people have seen the video and by 10:00 am on that same Monday, $10,000$ people have seen video.

(a) Find b and k. Round k to two decimal places.

Solution: *To find the values of* b *and* k*, we use the data. Measuring time in hours, and starting time (*$t = 0$*) at 1:00 am on Monday, we have*

$$N(0) = 1000 = \frac{1000000}{1 + be^0} = \frac{1000000}{1 + b} \implies 1 + b = \frac{1000000}{1000} = 1000 \implies \boxed{b = 999},$$

so

$$N(t) = \frac{1000000}{1 + 999e^{-kt}}.$$

Next, 10:00 am on the same day, is 9 *hours later, so*

$$10000 = N(9) = \frac{1000000}{1 + 999e^{-9k}} \implies 1 + 999e^{-9k} = \frac{1000000}{10000} = 100 \implies 999e^{-9k} = 99.$$

This means that

$$e^{-9k} = \frac{99}{999} \implies -9k = \ln\left(\frac{11}{111}\right) \implies \boxed{k = -\frac{1}{9}\ln\left(\frac{11}{111}\right) \approx 0.26}.$$

(b) How many people will have seen the video by 11:00 pm on that same day?

Solution: *11:00 pm on the same day is* 22 *hours after 1:00 am, so we want to find* $N(22)$*:*

$$N(22) \approx \frac{1000000}{1 + 999e^{-0.26\cdot 22}} \approx 233,840.$$

Comment: *In this case, using the rounded version of* k *leads to a more significant rounding error. If we use a less-rounded version of* k *(e.g., the full accuracy of my calculator), we would find that* $N(22) \approx 221,647$*. More evidence that,* ***unless instructed otherwise****, you should only round your final answer.*

(c) At what time (and day) will 900,000 people have seen the video?

Solution: *For this question, we solve the equation* $N(t) = 900,000$ *for the variable* t*:*

$$900,000 = N(t) = \frac{1000000}{1 + 999e^{-0.26t}} \implies 1+999e^{-0.26t} = \frac{1000000}{900000} \implies 999e^{-0.26t} = \frac{10}{9} - 1 = \frac{1}{9}.$$

Continuing, we have

$$e^{-0.26t} = \frac{1}{999}\cdot\frac{1}{9} \implies -0.26t = \ln\left(\frac{1}{8991}\right) \implies t = -\frac{1}{0.26}\ln\left(\frac{1}{8991}\right) \approx 35.$$

Conclusion: The number of people who have viewed the video will reach 900,000 *about* 35 *hours after 1:00 am on Monday, so at approximately noon on Tuesday.*

Comment: *If we were to use a less-rounded version of* k *for this part of the problem, we would find that* $t \approx 44.78$*, so our answer here is off by about* 30 *minutes.*

11. The wooden handle of a primitive axe found at an archaeological dig contains 30% of its initial amount of carbon-14. How old is the axe handle? You may assume that the half life of carbon-14 is 5730 years.

Solution: *Carbon-*14 *decays exponentially according to the rule* $C(t) = C_0e^{-kt}$*, where* $C(t)$ *is the amount of carbon at time* t *(in years) and* C_0 *is the initial amount of carbon. The exponent* k *can be found from the half-life by solving the equation*

$$e^{-k\cdot 5730} = \frac{1}{2} \implies -k\cdot 5730 = \ln(1/2) \implies k = -\frac{\ln(1/2)}{5730} \approx 0.000121.$$

Carbon-14 in organic compounds begins to decay when the organism, wood in this case, dies (i.e., when the tree the axe handle was made from is chopped down). Only 30% *of the* *original amount of C-14 was present in the axe handle when it was found, so if the tree was chopped down t years ago, it follows that*

$$C(t) = \cancel{C_0} e^{-0.000121t} = 0.3\cancel{C_0} \implies e^{-0.000121t} = 0.3 \implies -0.000121t = \ln 0.3$$

and therefore the axe handle is

$$t \approx -\frac{\ln 0.3}{0.000121} \approx 9950 \text{ years old.}$$