1. Find the polynomial of smallest degree whose graph matches the one in Figure 2, below.

Solution: This graph has $x$ intercepts $-2$, $1$ and $3$, so the polynomial we are looking for has factors $(x + 2)$, $(x - 1)$ and $(x - 3)$. The graph crosses the $x$ axis at $-2$ and $3$, so the factors $(x + 2)$ and $(x - 3)$ appear with odd powers, and the graph touches the $x$ axis at $1$, so the factor $(x - 1)$ appears with an even power. Since we want the polynomial to have the smallest degree possible, we choose the smallest powers possible for the three factors and conclude that

$$p(x) = a(x + 2)(x - 1)^2(x - 3),$$

for some constant $a$. To find $a$ we use the point $(2, -0.4)$ on the graph. We have

$$-0.4 = p(2) = a(2 + 2)(2 - 1)^2(2 - 3) = a \cdot (-4) \implies a = \frac{-0.4}{-4} = 0.1,$$

so

$$p(x) = 0.1(x + 2)(x - 1)^2(x - 3).$$

2. Follow steps 1 through 5 on page 205 of the textbook to analyze the polynomial

$$f(x) = 0.5(x + 1)(x - 2)^3.$$ 

Solution:

(a) End behavior: $f(x) = 0.5(x + 1)(x - 2)^3 = 0.5x^4 - 2.5x^3 + 3x^2 + 2x - 4$ has even degree and a positive leading coefficient, so $f(x) \to +\infty$ on both ends.
(b) **Intercepts:** The x intercepts are \(x = -1\) and \(x = 2\) and the y intercept is \(-4\).

(c) **Zeros:** The zeros of \(f(x)\) are \(x_1 = -1\) (multiplicity 1) and \(x_2 = 2\) (multiplicity 3). Since the multiplicities of both zeros are odd, the graph crosses the x axis at both zeros.

(d) **The graph has at most** \(4 - 1 = 3\) **turning points.**

(e) **Graph:** Using the points we have, the end behavior and the behavior (crossing) at the zeros, we can draw an initial, partial sketch:

\[
\begin{align*}
    &\text{If you simply connect the dots you will get something that looks like a parabola:} \\
    &\text{However this isn’t quite right, because it misses the behavior around the point } x = 2. \\
    &\text{The factor } (x - 2) \text{ appears with multiplicity 3, and because of this, the graph of } f(x) \text{ around the point } (2,0) \text{ will look like the graph of } y = x^3 \text{ around } (0,0):}
\end{align*}
\]
Combining the look of the graph around $x = 2$ with the earlier information, we get the graph:

3. Find the rational zeros of the polynomial

$$p(x) = 6x^3 - 17x^2 - 5x + 6.$$ 

**Solution:** Recall that if $k/l$ is a rational zero of a polynomial $f(x) = a_nx^n + \cdots + a_1x + a_0$ with integer coefficients, then $k$ must be factor of the constant coefficient $a_0$ and $l$ must be a factor of the leading coefficient $a_n$.

This means that if $k/l$ is a zero of $p(x)$, above, then $k$ and $l$ must both be factors of 6, i.e., $k = \pm 1, \pm 2, \pm 3$ or $\pm 6$ and $l = 1, 2, 3$ or 6 (we only need to $\pm$ on the numerator). This means that we need to check the rational numbers,

$$\frac{k}{l} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{3}{2} \text{ and } \pm \frac{2}{3}.$$ 

(Can you see why there are no others?)
Checking: \( p(1) = -10 \neq 0, p(-1) = -12 \neq 0, p(2) = -24 \neq 0, p(-2) = -100 \neq 0, p(3) = 0 \)!

Now, to save time (before checking the other 13 candidates) we divide \( p(x) \) by the factor \((x - 3)\):

\[
\begin{array}{c|c}
6x^2 + x - 2 & 6x^3 - 17x^2 - 5x + 6 \\
\hline
x - 3 & -6x^3 + 18x^2 \\
\hline
& x^2 - 5x \\
& -x^2 + 3x \\
& -2x + 6 \\
& 2x - 6 \\
& 0
\end{array}
\]

This means that \( 6x^3 - 17x^2 - 5x + 6 = (x - 3)(6x^2 + x - 2) \), and any remaining zeros of \( p(x) \) will also be zeros of \( p_1(x) = 6x^2 + x - 2 \). At this point, we can repeat the process above for \( p_1(x) \) — find the (shorter) list of candidates and check — or we can save time by using the quadratic formula and seeing if the zeros of \( p_1(x) \) are rational:

\[
-1 \pm \sqrt{1 + 4 \cdot 2 \cdot 6} \over 12 = 1 \pm 5 \over 12 = \frac{1}{2} \text{ or } \frac{-1}{3}.
\]

Conclusion: the rational zeros of \( p(x) \) are \( 3, \frac{1}{2} \) and \(-1/3\).

4. Consider the rational function:

\[
R(x) = \frac{x^2 - x - 2}{2x^2 + 5x - 3}.
\]

(a) Factor the numerator and denominator of \( R(x) \). What is the domain of this function?

Solution:

\[
x^2 - x - 2 = (x - 2)(x + 1) \quad \text{and} \quad 2x^2 + 5x - 3 = (2x - 1)(x + 3).
\]

The domain of \( R(x) \) is \( \{x | x \neq -3 \text{ and } x \neq 1/2\} \).

(b) Find the \( x \) and \( y \) intercepts of \( R(x) \) and determine the behavior of the function at each \( x \) intercept (‘crossing’ or ‘touching’).

Solution: The \( x \) intercepts are the zeros of the numerator, \( x_1 = -1 \) and \( x_2 = 2 \), and the \( y \) intercept is \( R(0) = 2/3 \). Since the factors \((x + 1)\) and \((x - 2)\) both appear to an odd power in the numerator, the graph \( y = R(x) \) crosses the \( x \) axis at these points. So far, our graph looks something like the figure below, though the directions the graph is crossing the \( x \) axis at the intercepts might be wrong.
(c) Find the vertical asymptotes of \( R(x) \) and determine the behavior of \( R(x) \) on either side of each vertical asymptote.

**Solution:** The vertical asymptotes occur at the zeros of the denominator, so they are the lines \( x = -3 \) and \( x = 1/2 \). The sign of \(+/-\) of \( R(x) \) can only change at zeros or at the vertical asymptotes, and since \( R(0) = 2/3 > 0 \), it follows that \( R(x) > 0 \) to the left of the asymptote \( x = 1/2 \), so \( R(x) \to \infty \) on the left side of the asymptote. Similarly, since \( R(1) = -1/2 < 0 \), it follows that \( R(x) \to -\infty \) on the right side of \( x = 1/2 \).

For the asymptote \( x = -3 \), we have \( R(-2) = -4/5 < 0 \) so \( R(x) \to -\infty \) to the right of \( x = -3 \), and since \( R(-4) = 2 > 0 \) it follows that \( R(x) \to \infty \) to the left of \( x = -3 \). Including this information we can improve the sketch of the graph to the one below:

(d) Find the horizontal or oblique asymptote of \( R(x) \), or say why no such line exists. If there is a horizontal/oblique asymptote, find any point(s) of intersection of the graph of \( R(x) \) and the asymptote.

**Solution:** Since the degrees of the numerator and denominator of \( R(x) \) are equal, and the ratio of their leading coefficients is \( \frac{1}{2} \), the line \( y = \frac{1}{2} \) is a horizontal asymptote to the graph. The graph of \( R(x) \) and this asymptote intersect at the point where \( R(x) = \frac{1}{2} \):

\[
\frac{x^2 - x - 2}{2x^2 + 5x - 3} = \frac{1}{2} \implies 2x^2 - 2x - 4 = 2x^2 + 5x - 3 \implies -7x = 1 \implies x = -\frac{1}{7}
\]
(e) Find the intervals where $R(x) > 0$ and $R(x) < 0$.

**Solution:** We did most of the work for this back in part (c), but I will summarize the results again here using the fact that the sign (+/−) of this function can only change at a zero or at vertical asymptote. This means that the sign of $R(x)$ is constant on the intervals:

$$(-\infty, -3), (-3, -1), (-1, 1/2), (1/2, 2) \text{ and } (2, \infty).$$

We sample a point in each interval, and the sign of the function at that point is the sign of the function in the interval.

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -3)$</th>
<th>$(-3, -1)$</th>
<th>$(-1, 1/2)$</th>
<th>$(1/2, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$-4$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$3$</td>
</tr>
<tr>
<td>$R(x)$</td>
<td>$2$</td>
<td>$-4/5$</td>
<td>$2/3$</td>
<td>$-1/2$</td>
<td>$2/15$</td>
</tr>
<tr>
<td>$Sign$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>point on graph</td>
<td>$(-4, 2)$</td>
<td>$(-2, -4/5)$</td>
<td>$(0, 2/3)$</td>
<td>$(1, -1/2)$</td>
<td>$(3, 2/15)$</td>
</tr>
</tbody>
</table>

Additionally, we can use the extra points to help plot the graph.

(f) Use all the information you have found to sketch the graph of $R(x)$. 

![Graph of R(x)](image)
5. For the functions

\[ f(x) = \frac{1}{x + 2} \quad \text{and} \quad g(x) = x^2 - 1 \]

find the composite functions

\[ f \circ f, \quad f \circ g, \quad g \circ f \quad \text{and} \quad g \circ g \]

and find the domain of each one.

**Solution:**

• \( f \circ f = \frac{1}{x + 2} + 2 = \frac{1}{x + 2} + \frac{2(x + 2)}{x + 2} = \frac{x + 2 + 2(x + 2)}{x + 2} = \frac{x + 2 + 2x + 4}{x + 2} = \frac{3x + 6}{x + 2} \]
  \[ \text{Domain: } \{ x | x \neq -2, -5/2 \} \]

• \( f \circ g = \frac{1}{(x^2 - 1) + 2} = \frac{1}{x^2 + 1} \]
  \[ \text{Domain: all } x. \]

• \( g \circ f = \left( \frac{1}{x + 2} \right)^2 - 1 = \frac{1 - (x + 2)^2}{(x + 2)^2} \]
  \[ \text{Domain: } \{ x | x \neq -2 \} \]

• \( g \circ g = (x^2 - 1)^2 - 1 = x^4 - 2x^2 - 2 \]
  \[ \text{Domain: all } x. \]

6. Find the inverse functions of \( H(x) = \frac{x + 1}{2x + 3} \) and \( G(x) = 4x^3 - 1 \). Find the domains of both inverses.

**Solution:** To find the inverse of a function \( f(x) \), we solve \( y = f(x) \) for \( x \) in terms of \( y \), and then swap the \( x \) and \( y \).

\[ H(x) : y = \frac{x + 1}{2x + 3} \Rightarrow y(2x + 3) = x + 1 \Rightarrow 2yx + 3y = x + 1 \Rightarrow 2yx - x = 1 - 3y \]

\[ \Rightarrow x(2y - 1) = 1 - 3y \Rightarrow x = \frac{1 - 3y}{2y - 1} \Rightarrow H^{-1}(x) = \frac{-3x + 1}{2x - 1} \]

The domain of \( H^{-1}(x) \) is \( \{ x | x \neq 1/2 \} \).

\[ G(x) : y = 4x^3 - 1 \Rightarrow 4x^3 = y + 1 \Rightarrow x^3 = \frac{y + 1}{4} \Rightarrow x = \sqrt[3]{\frac{y + 1}{4}} \Rightarrow G^{-1}(x) = \sqrt[3]{\frac{x + 1}{4}} \]

The domain of \( G^{-1}(x) \) is all \( x \).

7. The surface area of an inflatable globe of radius \( r \) is

\[ S = 4\pi r^2. \]

The radius of the ball increases as a function of time \( t \) (in seconds), according to the rule \( r = 2t^{1/3} \). Find the surface area of the ball as a function of \( t \).

**Solution:** To express the surface area as a function of time, we need to find the composite function \( S \circ r \):

\[ S \circ r = 4\pi (2t^{1/3})^2 = 16\pi t^{2/3}. \]
8. According to U.S. Census bureau, the world’s (human) population in 2013 was 7.13 billion and was growing at a rate of 1.1% per year. E.g., in 2014, the population was 1.1% greater than in 2013, so it was

\[ 7.13 + (1.1\%) \cdot 7.13 = 7.13 \cdot 1.011 \approx 7.208 \text{ billion}. \]

(a) Assuming that the world maintains this rate of growth, find the function

\[ P(t) = \text{world’s population, in billions, } t \text{ years after 2013}. \]

**Solution:** \( P(t) = 7.13(1.011)^t. \)

(b) According to the model you found in (a), what will the world’s population be in 2050? Round your answer to the nearest million.

**Solution:** *The year 2050 is 37 years after 2013, so the population in 2050 will be*

\[ P(37) = 7.13(1.011)^{37} = 10.68765\ldots \text{ billion.} \]

*Since we want to round to the nearest million, we round the answer to three decimal places. I.e., the population in 2050 will be about 10.688 billion or 10 billion, 688 million.*

(c) According to this model, in what year will the world’s population reach 20 billion?

**Solution:** *We want to find the value of } t \text{ such that } P(t) = 20…*

\[ 7.13(1.011)^t = 20 \implies (1.011)^t = \frac{20}{7.13} \implies \ln((1.011)^t) = \ln \left( \frac{20}{7.13} \right) \]

\[ \implies t \ln(1.011) = \ln \left( \frac{20}{7.13} \right) \implies t = \frac{\ln (20/7.13)}{\ln(1.011)} \approx 94.28 \]

*I.e., the population will reach 20 billion a little more than 94 years after 2013, so in the year 2107.*

9. Newton’s law of cooling/heating (see section 4.8) states that the temperature \( u(t) \) at time \( t \) of a body immersed in a medium of constant ambient temperature \( T \), can be modeled by the function

\[ u(t) = T + (u_0 - T)e^{kt}, \]

where \( u_0 \) is the initial temperature of the body (at the time of immersion) and \( k < 0 \) is a constant related to the heat conduction properties of the body.

A metal ball is heated to 500° Celsius and then immersed in a vat of ice water, that is kept at 0° Celsius. After 10 minutes the temperature of the ball is 400° Celsius. When will the ball reach a temperature of 100° Celsius?
Solution: The initial temperature of the ball is \( u_0 = 500 \) and the ambient temperature is \( T = 0 \), so according to Newton’s law, the temperature of the ball at time \( t \) (measured in minutes here) is

\[
u(t) = 0 + (500 - 0)e^{kt} = 500e^{kt}.
\]

To find the value of \( k \), we use the second data point: \( u(10) = 400 \):

\[
400 = u(10) = 500e^{k \cdot 10} \implies e^{10k} = \frac{400}{500} \implies 10k = \ln 0.8 \implies k = \frac{1}{10} \ln 0.8 \approx -0.0223,
\]

so that

\[
u(t) = 500e^{\left(\frac{1}{10} \ln 0.8\right)t}.
\]

To find the time when the temperature will be 100, we solve the equation \( u(t) = 100 \) for \( t \):

\[
500e^{\left(\frac{1}{10} \ln 0.8\right)t} = 100 \implies e^{\left(\frac{1}{10} \ln 0.8\right)t} = \frac{100}{500} \implies \left(\frac{1}{10} \cdot \ln 0.8\right)t = \ln 0.2 \implies t = \frac{10 \ln 0.2}{\ln 0.8}
\]

Conclusion: The ball will reach a temperature of 100° Celsius after \( t \approx 72.126 \) minutes.

Comment: I didn’t use the rounded (trimmed) version of \( k \) in the last calculation — I only rounded my final answer. If we use \( k \approx -0.0223 \) in the calculation above, we would find that the ball reached 100 degrees after \( t \approx 17.172 \) minutes. This is a difference of about 2.76 seconds. Not a big difference, but some cases, we might want the extra precision.

10. A video spreads virally on a certain social media platform according to the model

\[
N(t) = \frac{1,000,000}{1 + be^{-kt}},
\]

(the logistic model), where \( N(t) \) is the number of people who have seen the video, and the parameters \( b \) and \( k \) are (currently) unknown. At 1:00 am on Monday, 1000 people have seen the video and by 10:00 am on that same Monday, 10,000 people have seen video.

(a) Find \( b \) and \( k \). Round \( k \) to two decimal places.

Solution: To find the values of \( b \) and \( k \), we use the data. Measuring time in hours, and starting time (\( t = 0 \)) at 1:00 am on Monday, we have

\[
N(0) = 1000 = \frac{1000000}{1 + be^0} = \frac{1000000}{1 + b} \implies 1 + b = \frac{1000000}{1000} = 1000 \implies b = 999.
\]

so

\[
N(t) = \frac{1000000}{1 + 999e^{-kt}}.
\]

Next, 10:00 am on the same day, is 9 hours later, so

\[
10000 = N(9) = \frac{1000000}{1 + 999e^{-9k}} \implies 1 + 999e^{-9k} = \frac{1000000}{1000} = 100 \implies 999e^{-9k} = 99.
\]
This means that  

\[ e^{-9k} = \frac{99}{999} \implies -9k = \ln \left( \frac{11}{111} \right) \implies k = -\frac{1}{9} \ln \left( \frac{11}{111} \right) \approx 0.26. \]

(b) How many people will have seen the video by 11:00 pm on that same day?  
**Solution:** 11:00 pm on the same day is 22 hours after 1:00 am, so we want to find \( N(22) \):  

\[ N(22) \approx \frac{1000000}{1 + 999e^{-0.26 \cdot 22}} \approx 233,840. \]

**Comment:** In this case, using the rounded version of \( k \) leads to a more significant rounding error. If we use a less-rounded version of \( k \) (e.g., the full accuracy of my calculator), we would find that \( N(22) \approx 221,647 \). More evidence that, unless instructed otherwise, you should only round your final answer.

(c) At what time (and day) will 900,000 people have seen the video?  
**Solution:** For this question, we solve the equation \( N(t) = 900,000 \) for the variable \( t \):  

\[ 900,000 = N(t) = \frac{1000000}{1 + 999e^{-0.26t}} \implies 1 + 999e^{-0.26t} = \frac{1000000}{900000} \implies 999e^{-0.26t} = \frac{10}{9} - 1 = \frac{1}{9}. \]

Continuing, we have  

\[ e^{-0.26t} = \frac{1}{999} \cdot \frac{1}{9} \implies -0.26t = \ln \left( \frac{1}{8991} \right) \implies t = -\frac{1}{0.26} \ln \left( \frac{1}{8991} \right) \approx 35. \]

**Conclusion:** The number of people who have viewed the video will reach 900,000 about 35 hours after 1:00 am on Monday, so at approximately noon on Tuesday.  

**Comment:** If we were to use a less-rounded version of \( k \) for this part of the problem, we would find that \( t \approx 44.78 \), so our answer here is off by about 30 minutes.

11. The wooden handle of a primitive axe found at an archaeological dig contains 30% of its initial amount of carbon-14. How old is the axe handle? You may assume that the half life of carbon-14 is 5730 years.  
**Solution:** Carbon-14 decays exponentially according to the rule \( C(t) = C_0e^{-kt} \), where \( C(t) \) is the amount of carbon at time \( t \) (in years) and \( C_0 \) is the initial amount of carbon. The exponent \( k \) can be found from the half-life by solving the equation  

\[ e^{-k \cdot 5730} = \frac{1}{2} \implies -k \cdot 5730 = \ln(1/2) \implies k = -\frac{\ln(1/2)}{5730} \approx 0.000121. \]

Carbon-14 in organic compounds begins to decay when the organism, wood in this case, dies (i.e., when the tree the axe handle was made from is chopped down). Only 30% of the
original amount of C-14 was present in the axe handle when it was found, so if the tree was chopped down \( t \) years ago, it follows that

\[
C(t) = C_0 e^{-0.000121t} = 0.3C_0 \implies e^{-0.000121t} = 0.3 \implies -0.000121t = \ln 0.3
\]

and therefore the axe handle is

\[
t \approx -\frac{\ln 0.3}{0.000121} \approx 9950 \text{ years old.}
\]