1. Consider the function $k(x) = 3\sin\left(\frac{\pi}{4}(x-2)\right) + 1$. What is the period, amplitude and average value of $k(x)$? What are the maximum and minimum values of $k(x)$ and at which points do the minimum and maximum values occur? Use this information to sketch a graph of the function.

Solution: *The frequency of this function is $\omega = \pi/4$, so its period is*

$$T = \frac{2\pi}{\pi/4} = 8.$$

(*) The amplitude of function is $A = 3$, the average height is $b = 1$ and the horizontal shift is 2 (to the right).

(*) The maximum value attained by $k(x)$ is $k_{max} = 3 \cdot 1 + 1 = 4$ and the minimum value of $k(x)$ is $k_{min} = 3 \cdot (-1) + 1 = -2$.

(*) The maximum values of $y_1 = 3\sin x + 1$ occur at the points $x = \pi/2 + 2n\pi$ (n any integer, positive or negative or zero), so the maximum values of $y_2 = 3\sin((\pi/4)x) + 1$ occur at the points $x = 2 + 8n$ (n any integer). Finally, the maximum values of $k(x) = 3\sin\left(\frac{\pi}{4}(x-2)\right) + 1$ occur 2 to the right of the maximum values of y_2, i.e., they occur at the points $x = 4 + 8n$.

(*) In the same way, we find that the minimum values of $k(x)$ occur at the points $x = 8+8n$.

(*) Graph:

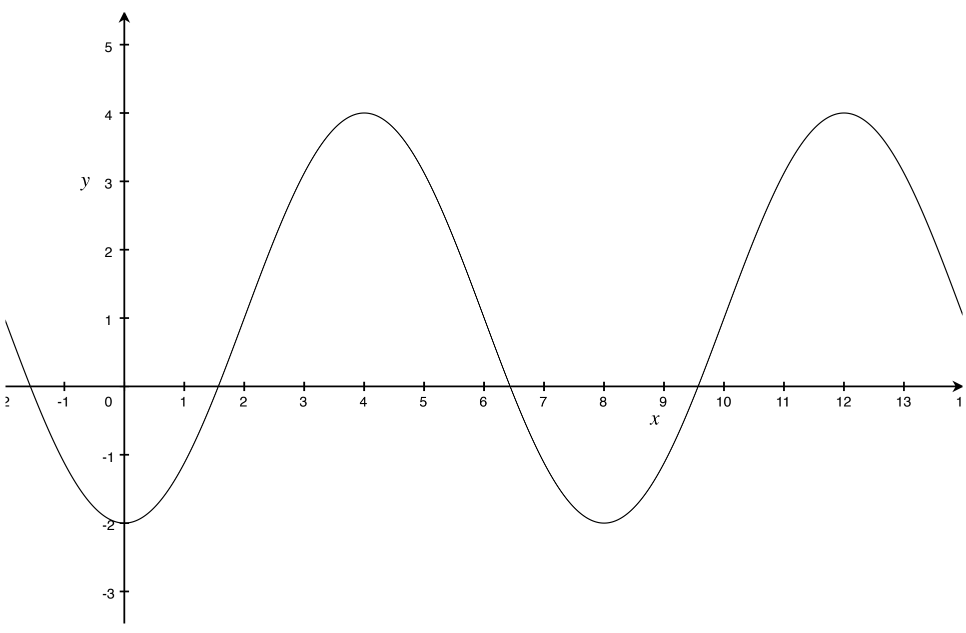


Figure 1: Graph of $y = k(x)$.

2. *Constructing a window.* A window is in the shape of a rectangle with a semicircular top, as in Figure 2, below. The perimeter of the window is to be 10 feet — what should the dimensions of the window be to maximize its area?

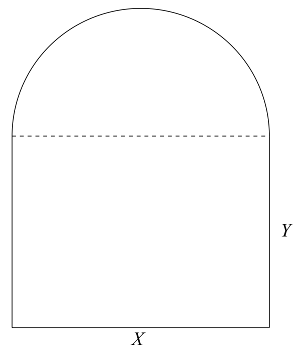


Figure 2: Window for problem 2.

Solution: *First, observe that the radius of the half circle is* $r = X/2$*, so the area of the half-circle is* $A_c = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi (X/2)^2 = \pi X^2/8$*. The area of the rectangular portion is* XY*, so the area of the entire window is*

$$A = XY + A_c = XY + \frac{\pi X^2}{8}.$$

Next, we use the information about the perimeter to express Y *in terms of* X*. The perimeter of the entire window is*

$$P = X + 2Y + P_c$$

where P_c *is the perimeter of the half-circle. Since the radius of the half-circle is* $r = X/2$*, its perimeter is*

$$P_c = \frac{1}{2} \cdot 2\pi r = \frac{\pi X}{2},$$

and therefore the perimeter of the entire window is

$$P = X + 2Y + \frac{\pi X}{2} = 2Y + \left(\frac{2+\pi}{2}\right) X.$$

Setting P *equal to* 10*, we find a relation between* X *and* Y*:*

$$2Y + \left(\frac{2+\pi}{2}\right) X = 10 \implies 2Y = 10 - \left(\frac{2+\pi}{2}\right) X \implies \boxed{Y = 5 - \left(\frac{2+\pi}{4}\right) X}$$

Now, we can express the area of the window as a function of X *alone:*

$$A = XY + \frac{\pi X^2}{8} = X\left[5 - \left(\frac{2+\pi}{4}\right) X\right] + \frac{\pi X^2}{8} = 5X - \left(\frac{\pi+4}{8}\right) X^2 = X\left(5 - \left(\frac{\pi+4}{8}\right) X\right).$$

The zeros of this function occur at $X = 0$ *and*

$$X = \frac{5}{\frac{\pi+4}{8}} = \frac{40}{\pi+4},$$

and the maximum value of the (quadratic) area function A *occurs halfway between the two zeros, at the point*

$$X_{max} = \frac{20}{\pi+4}.$$

Conclusion: *The dimensions of the window that maximize the area are*

$$X_{max} = \frac{20}{\pi + 4} \ (\approx 2.8 \textit{ feet}) \quad \textit{and} \quad Y_{max} = 5 - \left(\frac{\pi+2}{4}\right)\left(\frac{20}{\pi+4}\right) = \frac{10}{\pi+4} \ (\approx 1.4 \textit{ feet}).$$

The maximum area is

$$A_{max} = \frac{20}{\pi+4} \cdot \frac{10}{\pi+4} + \frac{\pi}{8} \cdot \left(\frac{20}{\pi+4}\right)^2 = \frac{200 + 50\pi}{(\pi+4)^2} \quad (\approx 7 \textit{ ft}^2)$$

3. Consider the rational function:

$$R(x) = \frac{x^3 + x^2 - 9x - 9}{2x^2 - 4x - 16}.$$

(a) Factor the numerator and denominator of $R(x)$. (Hint: $x^3 + x^2 = x^2(x+1)$).

$$R(x) = \frac{x^2(x+1) - 9(x+1)}{2(x^2 - 2x - 8)} = \frac{(x^2-9)(x+1)}{2(x+2)(x-4)} = \frac{(x-3)(x+3)(x+1)}{2(x+2)(x-4)}.$$

(b) Find the x and y intercepts of $R(x)$ and determine the behavior of the function at each x intercept ('crossing' or 'touching').

The x-intercepts are $(-3, 0)$, $(-1, 0)$ and $(3, 0)$. The graph ***crosses*** *at each one. The y-intercept is $(0, 9/16)$.*

(c) Find the vertical asymptotes of $R(x)$.

The vertical asymptotes are $x = -2$ and $x = 4$.

(d) Find the intervals where $R(x) > 0$ and $R(x) < 0$. Use this information to determine the behavior of $R(x)$ on either side of the vertical asymptotes.

The function can only change sign at zeros and points where it is undefined, so we need to evaluate the function in the intervals

$$(-\infty, -3), \ (-3, -2), \ (-2, -1), \ (-1, 3), \ (3, 4) \ \textit{and} \ (4, \infty).$$

I will collect the information in a 'sign-table':

Interval	*Point in interval*	*Value of function*	*Sign*	*Point on graph*
$(-\infty, -3)$	-5	$R(-5) = -32/27$	$-$	$(-5, -32/27)$
$(-3, -2)$	-2.5	$R(-2.5) = 33/52$	$+$	$(-2.5, 33/52)$
$(-2, -1)$	-1.5	$R(-1.5) = -27/44$	$-$	$(-1.5, -27/44)$
$(-1, 3)$	0	$R(0) = 9/16$	$+$	$(0, 9/16)$
$(3, 4)$	3.5	$R(3.5) = -117/44$	$-$	$(3.5, -117/44)$
$(4, \infty)$	6	$R(6) = 189/32$	$+$	$(6, 189/32)$

It follows that $R(x) \to \infty$ on the left of $x = -2$, $R(x) \to -\infty$ on the right of $x = -2$, $R(x) \to -\infty$ on the left of $x = 4$ and $R(x) \to \infty$ on the right of $x = 4$.

(e) Find the horizontal or oblique asymptote of $R(x)$. Find any point(s) of intersection of the graph of $R(x)$ and the asymptote. Describe the behavior of $R(x)$ as $x \to \infty$ and $x \to -\infty$.

The degree of the numerator, 3, is one more than the degree of the denominator, 2, which means that there is an ***oblique*** *asymptote to this graph. To find the equation of the asymptote, we divide the numerator by the denominator, with remainder:*

$$\begin{array}{r|l} & \tfrac{1}{2}x \quad + \tfrac{3}{2} \\ \hline 2x^2-4x-16 & x^3 \quad + x^2 - 9x \quad - 9 \\ & -x^3 + 2x^2 + 8x \\ \cline{2-2} & \qquad 3x^2 \quad - x \quad - 9 \\ & \quad\; -3x^2 + 6x + 24 \\ \cline{2-2} & \qquad\qquad 5x + 15 \end{array}$$

I.e.,

$$R(x) = \overbrace{\frac{1}{2}x + \frac{3}{2}}^{quotient} + \left(\frac{5x+15}{2x^2-4x-16}\right),$$

and the ***quotient*** *provides the equation of the oblique asymptote:*

$$y = \frac{1}{2}x + \frac{3}{2}.$$

This line has a positive slope, so $R(x) \to -\infty$ *as* $x \to -\infty$ *and* $R(x) \to \infty$ *as* $x \to \infty$.

(f) Use all the information you have found to sketch the graph of $R(x)$. Clearly mark intercepts and any other points of interest (e.g., the points you used to determine signs).

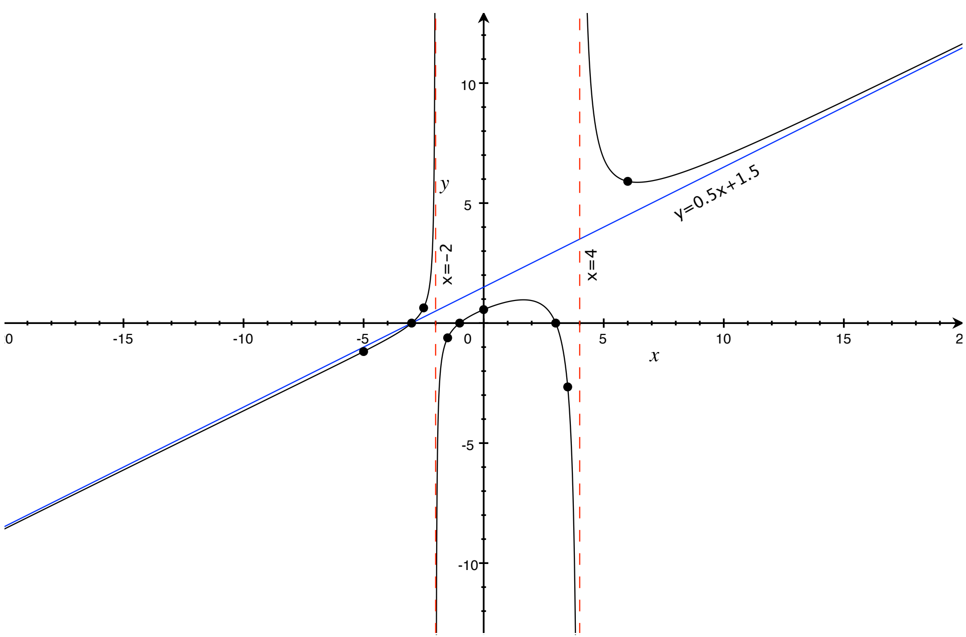


Figure 3: Graph of $R(x) = \dfrac{x^3+x^2-9x-9}{2x^2-4x-16}$.

4. Find the inverse functions of

$$f(x) = e^{3x-1} + 1, \ g(x) = \log_2(x^3+1) \ \text{and} \ h(x) = \sqrt[3]{4x-1} + 2$$

and find their domains.

To find inverse function of $F(x)$, we (i) solve the equation $x = F(y)$ for the variable y.

$f(x) = e^{3x-1} + 1$:

$$x = e^{3y-1} + 1 \implies x - 1 = e^{3y-1} \implies \ln(x-1) = 3y - 1 \implies y = \frac{1}{3}\ln(x-1) + \frac{1}{3},$$

i.e., $f^{-1}(x) = \frac{1}{3}\ln(x-1) + \frac{1}{3}$ and the domain of $f^{-1}(x)$ is $\{x | x > 1\}$.

$g(x) = \log_2(x^3 + 1)$:

$$x = \log_2(y^3 + 1) \implies 2^x = y^3 + 1 \implies y^3 = 2^x - 1 \implies y = \sqrt[3]{2^x - 1},$$

i.e., $g^{-1}(x) = \sqrt[3]{2^x - 1}$ and the domain of $g^{-1}(x)$ is all x.

$h(x) = \sqrt[3]{4x-1} + 2$:

$$x = \sqrt[3]{4y-1} + 2 \implies x - 2 = \sqrt[3]{4y-1} \implies 4y - 1 = (x-2)^3 \implies y = \frac{1}{4}(x-2)^3 + \frac{1}{4}$$

i.e., $h^{-1}(x) = \frac{1}{4}(x-2)^3 + \frac{1}{4}$ and the domain of $h^{-1}(x)$ is all x.

5. Sketch the graph of $h^{-1}(x)$ from problem 4. Indicate the x and y intercepts.

The graph of $h^{-1}(x)$ is obtained from the graph $y = x^3$ by (i) shifting up by 1, (ii) scaling (vertically) by 1/4...

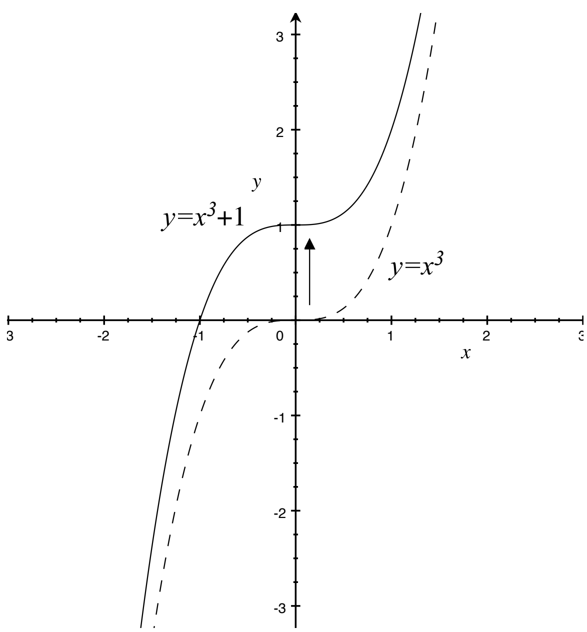
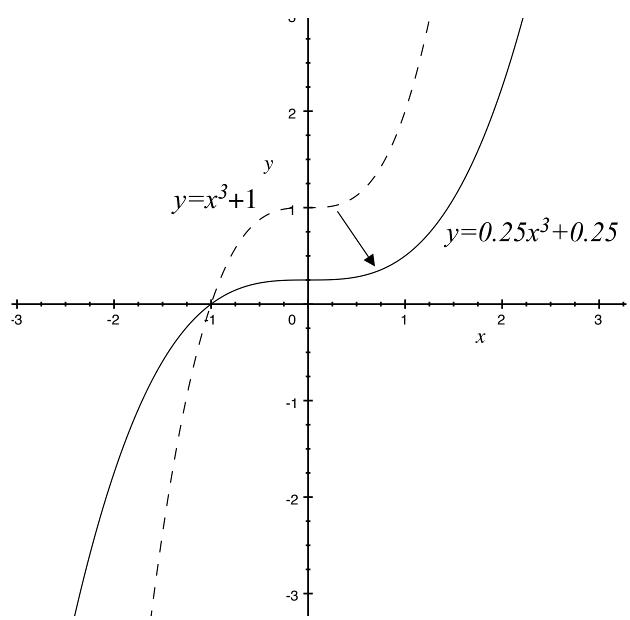


Figure 4: Shifting up by 1.



Scaling vertically by 1/4

and finally (iii) shifted right by 2:

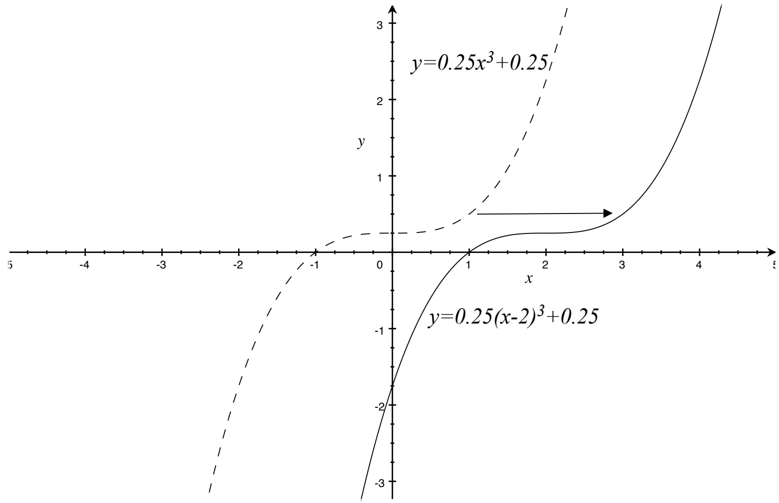


Figure 5: Graph of $h^{-1}(x) = 0.25(x-2)^3 + 0.25$

The x-intercept is $(1, 0)$ *and the y-intercept is* $(0, -1.75)$.

6. Simplify the following expressions, as described.

(a) Express

$$y = \frac{1}{3}\log_4(x^2+1) - 3\log_2(x-4)$$

as a single logarithm ***base* 2**. I.e., $y = \log_2(F(x))$.

First, from the change-of-base formula,

$$\log_4(x^2+1) = \frac{\log_2(x^2+1)}{\log_2(4)} = \frac{\log_2(x^2+1)}{2},$$

so that

$$y = \frac{1}{3}\log_4(x^2+1) - 3\log_2(x-4) = \frac{1}{3}\cdot\frac{\log_2(x^2+1)}{2} - 3\log_2(x-4) = \frac{1}{6}\log_2(x^2+1) - 3\log_2(x-4).$$

Next,

$$\frac{1}{6}\log_2(x^2+1) = \log_2\left((x^2+1)^{1/6}\right) \quad \textit{and} \quad 3\log_2(x-4) = \log_2\left((x-4)^3\right)$$

so that

$$y = \frac{1}{6}\log_2(x^2+1) - 3\log_2(x-4) = \log_2\left((x^2+1)^{1/6}\right) - \log_2\left((x-4)^3\right),$$

which means that

$$y = \log_2\frac{(x^2+1)^{1/6}}{(x-4)^3}.$$

(b) Express

$$w = \log\left(\frac{(x^3+5)^2(x-7)^3}{\sqrt[4]{2x+5}}\right)$$

as a sum of multiples of logarithms. I.e., $w = a\log(f(x)) + b\log(g(x)) + \cdots$, where the constants $a, b, \ldots$ can be positive or negative.

$$\Rightarrow \qquad w = 2\log(x^3+5) + 3\log(x-7) - \frac{1}{4}\log(2x+5).$$

7. The tides at Santa Cruz beaches exhibit periodic behavior. At high tide, which occurs at 8 am, the height of the water is 11.5 meters. At the subsequent low tide, which occurs at 2 pm, the height of the water is 8.5 meters.

(a) Find the *period, amplitude* and *average height* of the tides.

The time between a maximum value and the subsequent minimum value of a sinusoidal function is half a period, so the period is equal to twice the time between the max and next min. In this case, the time between the max and next min is 6 *hours, so the period of the tidal flow is* $T = 2 \cdot 6 = 12$ *hours.*

The amplitude of the tidal motion is half the distance between the max height and the min height, so in this case the amplitude is $A = \frac{1}{2}(11.5 - 8.5) = 1.5$.

Finally, the average height of the water is the average of the high and low tides, so $b = \frac{1}{2}(11.5 + 8.5) = 10$.

(b) Find the function $H(t) = b + A\cos(\omega(t - t_0))$, that gives the height (in meters) of the water at time t (in hours after midnight).

(c) *The frequency of the motion is* $\omega = 2\pi/T$, *so in this case, the frequency is* $\omega = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$. *With that information we know that*

$$H(t) = 10 + 1.5\cos\left(\frac{\pi}{6}(t - t_0)\right),$$

and it remains to find the horizontal shift, t_0.

An unshifted cosine function has a maximum value at 0, *and our function has a maximum at* 8 *hours after* 0 *(midnight* = 0 *hours), so the shift is* $t_0 = 8$, *and the height of the water at* t *hours after midnight is given by*

$$H(t) = 10 + 1.5\cos\left(\frac{\pi}{6}(t - 8)\right).$$

(d) What is the height of the water at 10 am? Do not use a calculator to find the answer.

The height of the water at 10 *am is*

$$H(10) = 10 + 1.5\cos\left(\frac{\pi}{6}(10 - 8)\right) = 10 + 1.5\cos(\pi/3) = 10 + 1.5 \cdot 0.5 = 10.75 \textit{ meters.}$$

8. The population of Tribbles on the Starship Enterprise is growing at a rate of 25% an hour. At 7 am on Monday, there are 20 Tribbles.

(a) How many Tribbles will there be at 8 am on Monday.

At 8 *am on Monday there will be* 25% *more Tribbles than there were at* 7 *am, so there will be* $20 + 0.25 \cdot 20 = 25$ *Tribbles at* 8 *am on Monday.*

(b) Find the function $T(t) = a \cdot b^t$ = number of Tribbles at time t (hours after 10 am on Monday). I.e., find a and b.

The number of Tribbles at time $t + 1$, *is* $T(t) + 0.25T(t) = T(t)(1.25)$. *So, at time* $t = 1$, *there are* $T(0)(1.25) = 20(1.25)$ *Tribbles, at time* $t = 2$ *there are*

$$T(2) = T(1)(1.25) = 20(1.25)(1.25) = 20(1.25)^2 \textit{ Tribbles,}$$

at time $t = 3$ *there are*

$$T(3) = T(2)(1.25) = 20(1.25)^2(1.25) = 20(1.25)^3 \text{ Tribbles,}$$

and so forth. I.e., at time $t > 0$ *there are*

$$T(t) = 20(1.25)^t \text{ Tribbles.}$$

(c) Express $T(t)$ in the form $T(t) = \alpha e^{\beta t}$. I.e., find α and β.

The natural logarithm of $(1.25)^t$ *is* $\ln(1.25)^t = t \ln 1.25$, *which means that*

$$(1.25)^t = e^{\ln(1.25)^t} = e^{(\ln 1.25)t},$$

so that

$$T(t) = 20(1.25)^t = 20e^{(\ln 1.25)t}.$$

(d) At what time on what day will the Tribble population reach 500? Round your answer to the nearest hour.

If $T(t) = 500$, *then*

$$20e^{\ln(1.25)\cdot t} = 500 \implies e^{\ln(1.25)\cdot t} = \frac{500}{20} = 25 \implies \ln(1.25)\cdot t = \ln 25 \implies t = \frac{\ln 25}{\ln 1.25} \approx 14.42.$$

I.e., the Tribble population will reach 500 *about* $14.42 \approx 14$ *hours after* 7 *am on Monday, which is* 9 *pm on the same Monday.*

9. Joe borrows \$500,000 from Mike, and agrees to pay the full amount back, plus interest in 10 years. The agree on an interest rate of $r = 4.5\%$ compounded monthly. How much will Joe have to pay Mike at the end of 10 years?. How much interest did he pay?

Since the interest is compounding monthly, Joe's debt to Mike after t *years is*

$$D(t) = 500,000\left(1 + \frac{0.045}{12}\right)^{12t},$$

which means that after $t = 10$ *years, Joe will have to pay Mike*

$$D(10) = \$500,000\left(1 + \frac{0.045}{12}\right)^{120} \approx \$783,496.$$

The total interest he paid is $\$783,496 - \$500,000 = \$283,496$.

10. Express the values of the following trigonometric functions as rational numbers, rational multiples of $\sqrt{2}$, $\sqrt{3}$ or $\sqrt{6}$ or sums/differences of numbers like that. (No calculators)

Comment: We will use the following 'well-known' values of the sine and cosine functions:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

We will also use the fact that cosine is an even function and sine is odd, and that they are both periodic with period 2π.

(a) $\sec(2\pi/3) =$

We know that $\cos(2\pi/3) = -1/2$, *so* $\sec(2\pi/3) = \dfrac{1}{\cos(2\pi/3)} = -2$.

(b) $\csc(3\pi/4) =$

We know that $\sin(3\pi/4) = \sqrt{2}/2$, *so* $\csc(3\pi/4) = \dfrac{1}{\sin(3\pi/4)} = \dfrac{1}{\sqrt{2}/2} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

(c) $\tan(-2\pi/3) =$

We know that

$$\cos(-2\pi/3) = \cos(2\pi/3) = -1/2$$

(because cos *is even), and*

$$\sin(-2\pi/3) = -\sin(2\pi/3) = -\sqrt{3}/2$$

(because sin *is odd), so*

$$\tan(-2\pi/3) = \frac{\sin(-2\pi/3)}{\cos(-2\pi/3)} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}.$$

(d) $\sin(7\pi/4) =$

Observe that $7\pi/4 - 2\pi = -\pi/4$, *so*

$$\sin(7\pi/4) = \sin(7\pi/4 - 2\pi) = \sin(-\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2.$$

(e) $\sin(\pi/12) =$

Observe that $\pi/12 = (\pi/4) - (\pi/6)$, *so using the identity*

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha,$$

we have

$$\begin{aligned}\sin(\pi/12) &= \sin((\pi/4) - (\pi/6)) \\ &= \sin(\pi/4)\cos(\pi/6) - \sin(\pi/6)\cos(\pi/4) \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\end{aligned}$$

11. A fossilized insect contains 20% of its initial amount of carbon-14. How many years ago did the insect die? You may assume that the half life of carbon-14 is 5730 years.

The amount of carbon-14 in the insect t *years after it died is given by the formula*

$$C(t) = C_0 \cdot e^{-rt},$$

where r is rate of decay of carbon-14 and C_0 is the initial amount of carbon-14. If 20% *of the initial amount is left after t years, then*

$$\cancel{C_0}e^{-rt} = 0.2\cancel{C_0} \implies e^{-rt} = 0.2 \implies -rt = \ln 0.2 \implies t = -\frac{\ln 0.2}{r}.$$

To find r, we use the half-life $t_h = 5730$, which is the amount of time it takes for half of the carbon-14 to decay, i.e.,

$$e^{-r\cdot 5730} = 0.5 \implies -r\cdot 5730 = \ln 0.5 = -\ln 2 \implies r = \frac{\ln 2}{5730} \quad (\approx 0.000121).$$

Therefore, the insect died (approximately)

$$t = -\frac{\ln 0.2}{\ln 2/5730} \approx 13305 \text{ years ago.}$$

12. Sketch the graphs of the following functions. For each one, find the domain, mark the x and y intercepts, if they exist, and indicate the behavior of the function as $x \to \pm\infty$ if appropriate.

(a) $f(x) = 2\ln(x+3)$.

This graph is a shift to the left by 3 units of $y = \ln x$, followed by vertical scaling by the factor 2. The y-intercept is $(0, 2\ln 3)$ and the x-intercept is the point where $\ln(x+3) = 0 \implies x+3 = 1 \implies x = -2$, so the point is $(-2, 0)$.
The domain of this function is $\{x|x > -3\}$ and $f(x) \to -\infty$ as $x \to -3$ (from the right). As $x \to \infty$, $f(x) \to \infty$.

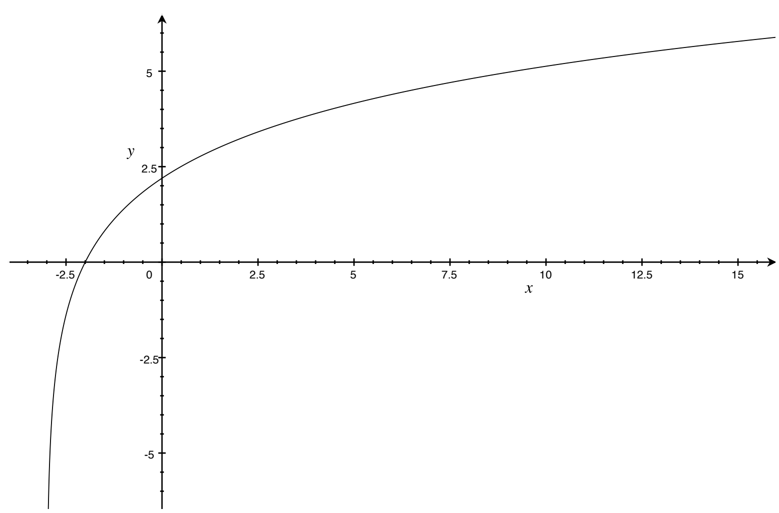


Figure 6: Graph of $f(x) = 2\ln(x+3)$

(b) $g(x) = 2^{x-3} - 4$

This function is defined for all x. As $x \to \infty$, $g(x) \to \infty$, and as $x \to -\infty$, $g(x) \to -4$, because $2^{x-3} \to 0$ as $x \to -\infty$. The y-intercept is $(0, g(0)) = (0, -31/40)$ and the x-intercept is the point where $g(x) = 0$:

$$2^{x-3} - 4 = 0 \implies 2^{x-3} = 4 \implies x - 3 = \log_2 4 = 2 \implies x = 2 + 3 = 5,$$

i.e., the x-intercept is $(5, 0)$. Its graph is obtained from the graph of $y = 2^x$ by shifting to the right by 3 and down by 5.

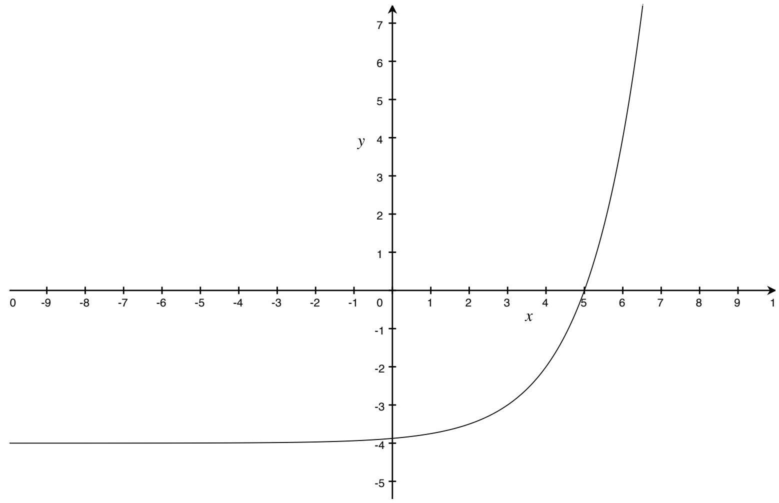


Figure 7: Graph of $g(x) = 2^{x-3} - 4$

13. Solve the equation $4^{x+2} = 2 \cdot 5^{x-1}$. Express your answer in terms of the natural logarithm function. Don't use a calculator.

First take natural logarithms of both sides,

$$\ln(4^{x+2}) = \ln(2 \cdot 5^{x-1}) \implies (x+2)\ln 4 = \ln 2 + (x-1)\ln 5$$

then simplify

$$\implies (\ln 4)x + 2\ln 4 = (\ln 5)x + (\ln 2 - \ln 5) \implies (\ln 5 - \ln 4)x = 2\ln 4 + \ln 5 - \ln 2$$

and solve for x (and simplify some more):

$$x = \frac{2\ln 4 + \ln 5 - \ln 2}{\ln 5 - \ln 4} \quad \left(= \frac{\ln(4^2 \cdot 5/2)}{\ln(5/4)} = \frac{\ln 40}{\ln 1.25} \right).$$

14. Solve the equation $7\sin^2 x + 3\cos^2 x = 6$. List all possible solutions.
(Hint: $\sin^2 x + \cos^2 x = 1$.)

First, we write

$$7\sin^2 x + 3\cos^2 x = 4\sin^2 x + 3(\sin^2 x + \cos^2 x) = 4\sin^2 x + 3,$$

so that

$$7\sin^2 x + 3\cos^2 x = 6 \implies 4\sin^2 x = 3 \implies \sin^2 x = \frac{3}{4} \implies \sin x = \pm\frac{\sqrt{3}}{2}.$$

*Now, we know that $\sin(\pi/3) = \sqrt{3}/2$ (quadrant I), $\sin(2\pi/3) = \sqrt{3}/2$ (quadrant II). And since $\sin x$ is an **odd** function, it follows that $\sin(-\pi/3) = -\sqrt{3}/2$ (quadrant IV), $\sin(-2\pi/3) = -\sqrt{3}/2$ (quadrant III). Therefore, taking the periodicity into account, the set of all solutions is*

$$x = \frac{\pi}{3} + 2n\pi, \quad \frac{2\pi}{3} + 2n\pi, \quad -\frac{\pi}{3} + 2n\pi \;\; \textit{and} \;\; -\frac{2\pi}{3} + 2n\pi,$$

where n is can be any integer. We can simplify this description by observing that

$$\frac{\pi}{3} - \pi = -\frac{2\pi}{3} \qquad \textit{and} \qquad \frac{2\pi}{3} - \pi = -\frac{\pi}{3},$$

so that the set of solutions is

$$\left\{ x \,\middle|\, x = \frac{\pi}{3} + n\pi \ \textit{or}\ x = \frac{2\pi}{3} + n\pi \right\},$$

where n can be any integer, positive, negative or zero.

15. Find the linear function $y = f(x)$ that passes through the points $(-1, 2)$ and $(2, 3)$.

First we find the slope of the graph $y = f(x)$:

$$m = \frac{3-2}{2-(-1)} = \frac{1}{3},$$

so $f(x) = \frac{1}{3}x + b$. To find the y-intercept, b, we use one of the two given points:

$$2 = f(-1) = \frac{1}{3} \cdot (-1) + b = b - \frac{1}{3} \implies b = \frac{7}{3}.$$

So, the linear function is

$$f(x) = \frac{1}{3}x + \frac{7}{3}.$$